SOME 1-COHOMOLOGY COMPUTATIONS FOR GROUPS OF LIE TYPE OF RANK 4 IN CHARACTERISTIC 2

BY

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This paper concerns the computation of the cohomology groups $H^1(G, M)$, where G is the finite group $SL_5(2^n)$, the finite group $SU_5(2^n)$, the finite group $Spin_8(2^n)$, the finite group $Spin_9(2^n)$, the (simply connected) algebraic group $A_4(\bar{F}_2)$, the (simply connected) algebraic group $B_4(\bar{F}_2)$, the (simply connected) algebraic group $C_4(\bar{F}_2)$, or the (simply connected) algebraic group $D_4(\bar{F}_2)$, and where M is a simple module. The cohomology group $H^1(G, M) \cong \mathrm{Ext}^1_G(k, M)$ can be thought of as a space of equivalence classes of extensions of M by the one-dimensional trivial module k. In the course of deriving the needed results concerning 1-cohomology over the algebraic groups, we take the opportunity to not only compute the quantities $\mathrm{Ext}^1_G(k, M)$ for all simple G-modules M, but to also compute all of the quantities $\mathrm{Ext}^1_G(L, M)$ for all pairs of simple G-modules (L, M). The strategy is to first determine the socle layers of the ext module for the Frobenius kernel of G by tensoring with various restricted simple modules, and using an isomorphism derived from the Lyndon-Hochschild-Serre spectral sequence to determine which modules appear in

the socle of the resulting tensor product We use the resulting information to compute all of the groups $\mathrm{Ext}^1_G(L, M)$ for the (simply connected) algebraic groups of type A_4 and D_4 using the 5-term sequence. We then use the Lyndon-Hochschild-Serre spectral sequence for the pair $(G, \tilde{G}_T)$, where $\tilde{G}_T$ is a certain infinitesimal subgroup of simply connected C_4, together with the information gathered about the ext groups for D_4, to calculate the ext groups for $\tilde{G}_T$. Finally, we combine this information with known results about the ext groups for G_σ (where G_σ is a certain infinitesimal subgroup of simply connected B_4) to calculate all of the ext groups for (simply connected) B_4 and C_4, and their Lie algebras.

The bulk of the argument for the finite groups involves the reduction of the problem to a reasonable finite number of cases where the cohomology might be nonzero. We show that the 1-cohomology groups vanish in a large number of cases by using a generalization of Alperin's induction step obtained from the long exact sequence in cohomology. In the case of the B_4-type groups, we are able to take advantage of a very simple consequence of the special isogeny that exists between the algebraic groups of type B_4 and C_4. In the course of many of the arguments involving the finite groups, we need to show that certain hom groups are zero, thus we need to develop a lot of information about which simple modules appear as composition factors of certain tensor products of simple modules. An important tool in this type of analysis will be the concept of module "mass", which was first introduced in the papers of Sin. Finally, we handle the remaining cases by using information about cohomology over the algebraic group, with a suitable bound on n, we may use the relationship between rational and generic cohomology, as documented by Cline, Parshall, Scott, Van der Kallen, and Andersen

CHAPTER 1

INTRODUCTION

In 1939, Hermann Weyl [23] observed that finite dimensional representations of complex and real Lie groups could always be written as direct sums of irreducible ones. The situation for Chevalley groups over fields of nonzero characteristic is different. For these representations (over the fields of the natural characteristic), not all modules are semisimple; there exist modules which have a simple submodule such that the quotient by this submodule is simple, and which cannot be written as a direct sum. These are sometimes referred to as "non split" extensions. In this paper, we classify all modules of composition length two for certain algebraic groups of rank 4 over an algebraically closed field of characteristic 2. We then utilize the results for modules in the principal block to assist in determining all of the extensions of the trivial module for most (all but finitely many) of the finite classical groups over characteristic 2 fields with rank 4 root systems. Specifically, we calculate the extensions of the trivial module for $\mathrm{Spin}_8(2^n)$ if $n > 9$, for $\mathrm{Spin}_9(2^n)$ if $n > 10$, and for $\mathrm{SL}_5(2^n)$ if $n > 11$.

The precise way of formulating the classification of module extensions is obtained via homological algebra, namely, by considering the first right derived functor of $\mathrm{Hom}_G(\ ,\ )$. Given two G-modules L and M, the (vector space) $\mathrm{Ext}^1_G(L, M)$ can be viewed as a space of equivalence classes of short exact sequences of the form

$$0 \longrightarrow M \longrightarrow A \longrightarrow L \longrightarrow 0,$$

where equivalence is defined by commutative diagrams of the form

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & A \arrow[r] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & B \arrow[r] \arrow[u, "\cong"] & L \arrow[r] & 0 \end{tikzcd}$$

The corresponding element of the space $\mathrm{Ext}^1_G(L, M)$ is zero if and only if the short exact sequence is split, i.e., if and only if

$$A \cong L \oplus M.$$

Thus, the dimension of the space $\mathrm{Ext}^1_G(L, M)$ in some sense quantifies the amount of variegation which can occur in the types of nonsplit extensions. More precisely, $\dim_k(\mathrm{Ext}^1_G(L, M))$ is the maximum number d of summands of (modules isomorphic to) M which can occur in a module with a filtration

$$\begin{array}{c} L \\ \bigoplus_{i=1}^{d} M, \end{array}$$

where L is the only quotient which can be written as a direct sum of simple modules. We shall consider the extension problem for the simply connected groups of types A_4, B_4, C_4 and D_4. (The case $G = F_4$ has already been investigated [19]; due to the existence of a certain endomorphism of G together with a refinement of Steinberg's tensor product theorem, that problem had turned out to be essentially of rank two.) We compute all of the groups $\mathrm{Ext}^1_G(L, M)$ for all possible pairs of simple modules L and M. In the process, we also obtain the structure of the G-modules $\mathrm{Ext}^1_{G_1}(L, M)$ for all possible pairs of simple restricted modules L and M, where G_1 is the Frobenius kernel and has the same representation theory as the Lie algebra $\mathrm{Lie}(G)$.

In 1942, Brauer [5] described how to compute the modular character tables for the finite group $SL(2, p)$ and determined the structure of the projective indecomposables. It was not until the late seventies that Alperin solved the extension problem for $SL(2, 2^n)$, and Cline published results for the algebraic groups $SL(2, k)$ over algebraically closed fields k of characteristic p (for all primes p). In 1977, Cline, Parshall,

Scott, and Van der Kallen published an important result describing the relationship between cohomology for the algebraic group and cohomology for the finite Chevalley groups. In 1986, Andersen published an analogous result for the twisted Chevalley groups. Work done by various algebraists in the eighties and early nineties, starting with Cline, Parshall, and Scott (e.g., Jones, Volklein, Kleschchev) produced results on the extension problem in some cases for arbitrary Chevalley group types over arbitrary fields but for very restricted classes of modules, and in other cases for specific Chevalley group types of small rank (e.g. $SL(3,k)$) over arbitrary fields (e.g., Kühne-Hausmann, Doty-Sullivan, Irving, Krop). During the years 1991-3, Sin produced several results on the extension problem for specific Chevalley group types of rank 2 and 3 over fields of characteristic 2 and 3, following Alperin's approach [1]. In 1993, Ye considered the groups $G_2(p)$ for $p > 7$. As far as the near future is concerned, the (imminent) proof of the Lusztig conjecture will yield results for arbitrary Chevalley groups, but only over fields of characteristic p for sufficiently large p (and it is not at all clear at present exactly how large p must be for the results to hold.) This paper utilizes the type of methods developed by Andersen [2], Jantzen [12] and Donkin [9,10] for the algebraic groups and Sin [15,16,17,18] for the finite groups.

We shall exploit some relationships which occur between the groups of types B_4, C_4, D_4 and F_4. If G and $\tilde{G}$ are simply connected groups of types B_l and C_l, defined over $\mathbb{F}_2$, then there are maps from G to $\tilde{G}$, and from $\tilde{G}$ back to G whose composites are the Frobenius morphisms of G and $\tilde{G}$. It turns out that a particular subgroup of G which is of isomorphism type simply connected D_4 plays a special role with respect to these maps. In this way, the group of type D_4 becomes the natural starting point for our considerations. A general discussion of this situation (for the groups of rank l) is presented in ref. 11 where the results of Chapters 3 to 5 will appear.

In recent work of Sin [19], similar isogenies for the groups of types B_2 and F_4 in

characteristic 2 and type G_2 in characteristic 3 were exploited to compute the simple module extensions. The kernel of the infinitesimal isogeny for F_4 also happens to be isomorphic to a quotient of a Lie algebra of type D_4, and this link made it possible to draw on knowledge about D_4 in making calculations about F_4. This time, in our calculations we shall transfer information in the opposite direction, from F_4 via D_4 to B_4 and C_4.

In Chapter 2, we outline the strategies that we will be using to compute the algebraic group extensions, and perform some preliminary computations concerning Weyl modules and tensor products. In Chapter 3, we compute the socles of all of the tensor products we will need in the rest of the paper. We need to do this first because some information about hom groups is required in the method of determining the structure of the Ext^1 groups in Chapter 4. In Chapter 5, we compute the module extensions for the algebraic group by using the information obtained in Chapters 3 and 4 together with the Lyndon-Hochschild-Serre spectral sequence. Finally, we consider the finite groups in Chapter 6. (Chapter 6 may be read independently although it requires some of the results of Chapters 4 and 5.) The preliminaries are as follows. Fix an algebraic closure k of $\mathbb{F}_2$, and regard finite extensions of $\mathbb{F}_2$ as subfields of k. For $n \in \mathbb{N}$, we denote by G the simply connected semisimple algebraic group of type D_4 or B_4 over k, and by $G(n)$ the finite group $\mathrm{Spin}_8(2^n)$, the finite group $\mathrm{Spin}_9(2^n)$, the finite group $\mathrm{SL}_8(2^n)$, or the finite group $\mathrm{SU}_8(2^n)$. The latter is by definition the subgroup of $\mathrm{SL}_4(2^{2n})$ preserving the hermitian form on $\mathbb{F}^4_{2^{2n}}$ represented in the standard basis by the identity matrix. Thus, $G(n)$ can always be regarded as the subgroup of fixed points under an appropriate endomorphism of G. Let T be a maximal torus of G, and for dominant weights $\mu \in X^+(T)$, with respect to a fixed choice of Borel subgroup containing T, let $L(\mu)$ denote the

unique (up to isomorphism) simple module for G with highest weight μ. For a module M, over G or $G(n)$, we denote by M^* its dual (contragredient). We denote by M_i, or occasionally $M^{(2^i)}$ (when dealing with the groups of types B_4 and C_4 and their special isogenies), the i^{th} Frobenius twist of M. The set of (isomorphism classes of) simple modules for $\mathrm{Spin}_8(2^n)$ (resp. $\mathrm{Spin}_9(2^n)$, $\mathrm{SL}_5(2^n)$, $\mathrm{SU}_5(2^n)$) is comprised of the restriction to $\mathrm{Spin}_8(2^n)$ (resp. $\mathrm{Spin}_9(2^n)$, $\mathrm{SL}_5(2^n)$, $\mathrm{SU}_5(2^n)$) of the 2^n-restricted modules for G. By Steinberg's tensor product theorem, this will be the restriction to $G(n)$ of the set of modules of the form $L(\mu_0) \otimes L(\mu_1)_1 \otimes \cdots \otimes L(\mu_n)_n \cong L(\mu_0) \otimes L(2\mu_1) \otimes \cdots \otimes L(2^n\mu_n) \cong L(\mu_0 + 2\mu_1 + \cdots + 2^n\mu_n)$, as $\mu_0, \ldots, \mu_n$ range over the restricted weights (i.e. those integral weights λ for which $0 \leq \langle \lambda, \alpha_i^\vee \rangle < 2$ for each simple root α_i). We note however, that $M_{i+n} \cong M_i$ if $G(n) = \mathrm{Spin}_8(2^n)$, $\mathrm{Spin}_9(2^n)$, or $\mathrm{SL}_5(2^n)$, while $M_{i+n} \cong M_i^*$ if $G(n) = \mathrm{SU}_5(2^n)$.

We label the modules corresponding to the restricted weights as follows. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote the standard fundamental dominant weights for a root system of type A_4, let $\delta_1, \delta_2, \delta_3, \delta_4$ denote the standard fundamental dominant weights for a root system of type D_4, let $\omega_1, \omega_2, \omega_3, \omega_4$ denote the standard fundamental dominant weights for a root system of type B_4, and let $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4$ denote the standard fundamental dominant weights for a root system of type C_4.

TABLE 1-1 (D_4 RESTRICTED MODULES)

symbol	module	dim	mass
Θ^a	$L(\delta_1)$	8	3
Θ^b	$L(\delta_3)$	8	3
Θ^c	$L(\delta_4)$	8	3
μ	$L(\delta_2)$	26	5
Λ^a	$L(\delta_2+\delta_4)$	48	6
Λ^b	$L(\delta_1+\delta_4)$	48	6
Λ^c	$L(\delta_1+\delta_3)$	48	6
Δ^a	$L(\delta_1+\delta_2)$	160	8
Δ^b	$L(\delta_2+\delta_3)$	160	8
Δ^c	$L(\delta_2+\delta_4)$	160	8
Ψ	$L(\delta_1+\delta_3+\delta_4)$	246	8
Γ^a	$L(\delta_2+\delta_3+\delta_4)$	784	11
Γ^b	$L(\delta_1+\delta_2+\delta_4)$	784	11
Γ^c	$L(\delta_1+\delta_2+\delta_3)$	784	11
S	$L(\delta_1+\delta_2+\delta_3+\delta_4)$	4096	14

Table 1-2 (A_4 restricted modules)

symbol	module	dim	mass
Θ	$L(\lambda_1)$	5	2
Λ	$L(\lambda_2)$	10	3
μ	$L(\lambda_1+\lambda_4)$	24	4
Δ	$L(\lambda_2+\lambda_4)$	40	5
Σ	$L(\lambda_3+\lambda_4)$	40	5
Ψ	$L(\lambda_2+\lambda_3)$	74	6
Υ	$L(\lambda_1+\lambda_2+\lambda_4)$	160	7
Γ	$L(\lambda_1+\lambda_2+\lambda_3)$	280	8
S	$L(\lambda_1+\lambda_2+\lambda_3+\lambda_4)$	1024	10

Table 1-3 (B_4 restricted modules)

symbol	module	dim	mass
Θ	$L(\omega_1)$	9	4
μ	$L(\omega_2)$	26	7
Λ	$L(\omega_3)$	48	9
Δ	$L(\omega_1+\omega_2)$	160	11
Ψ	$L(\omega_1+\omega_3)$	246	13
Γ	$L(\omega_2+\omega_3)$	784	16
Σ	$L(\omega_1+\omega_2+\omega_3)$	4096	20
σ	$L(\omega_4)$	16	5
S	$L(\omega_1+\omega_2+\omega_3+\omega_4)$	65536	25

Because of the special isogeny which exists between the algebraic groups of type B_4 and C_4, it turns out that for $G = B_4$, we have $L(\lambda) \otimes L(\omega_4) \cong L(\lambda+\omega_4)$, for $\lambda \in \{\omega_1, \omega_2, \omega_3, \omega_1+\omega_2, \omega_1+\omega_3, \omega_2+\omega_3, \omega_1+\omega_2+\omega_3\}$. (See ref 21.) Thus, we refer

to $L(\omega_1 + \omega_4)$ as $\Theta\sigma$, etc. The fact that S, the first Steinberg module, is actually a tensor product of two smaller modules will be of great help in some of the induction step arguments.

For a finite set I of natural numbers, we let $V_I = \bigotimes_{i\in I} V_i$. The collection of simple $kG(n)$-modules then consists of the set of all (isomorphism classes of) modules of the form

$$\Theta^a_{I_1}\otimes\Theta^b_{I_2}\otimes\Theta^c_{I_3}\otimes\mu_{I_4}\otimes\Lambda^a_{I_5}\otimes\Lambda^b_{I_6}\otimes\Lambda^c_{I_7}\otimes\Delta^a_{I_8}\otimes\Delta^b_{I_9}\otimes\Delta^c_{I_{10}}\otimes\Psi_{I_{11}}\otimes\Gamma^a_{I_{12}}\otimes\Gamma^b_{I_{13}}\otimes\Gamma^c_{I_{14}}\otimes S_R,$$

if $G = D_4$, the set of modules of the form

$$\Theta_{I_1}\otimes\Theta^*_{I_2}\otimes\Lambda_{I_3}\otimes\Lambda^*_{I_4}\otimes\mu_{I_5}\otimes\Delta_{I_6}\otimes\Delta^*_{I_7}\otimes\Sigma_{I_8}\otimes\Sigma^*_{I_9}\otimes\Psi_{I_{10}}\otimes\Upsilon_{I_{11}}\otimes\Upsilon^*_{I_{12}}\otimes\Gamma_{I_{13}}\otimes\Gamma^*_{I_{14}}\otimes S_R,$$

if $G = A_4$, or the set of modules of the form

$$\Theta_{I_1}\otimes\mu_{I_2}\otimes\Lambda_{I_3}\otimes\Delta_{I_4}\otimes\Psi_{I_5}\otimes\Gamma_{I_6}\otimes\Sigma_{I_7}\otimes\sigma_{I_8}\otimes(\Theta\sigma)_{I_9}\otimes(\mu\sigma)_{I_{10}}\otimes(\Lambda\sigma)_{I_{11}}\otimes(\Delta\sigma)_{I_{12}}$$

$$\otimes(\Psi\sigma)_{I_{13}}\otimes(\Gamma\sigma)_{I_{14}}\otimes S_R,$$

if $G = B_4$, where $I_1, \ldots, I_{14}, R$ are disjoint subsets of $N = \{0, 1, \ldots, n-1\}$. It is well-known that the module S_N is projective; it is the Steinberg module for $G(n)$. The group of field automorphisms $Gal(\mathsf{F}_{2^n}/\mathsf{F}_2)$ acts on the set of isomorphism classes of simple $kG(n)$ modules by acting on the set of ordered 15-tuples of disjoint subsets of N. The automorphism $\gamma \mapsto \gamma^{2^i}$ acts by adding i to each element of N and taking the remainder modulo n, if $G(n) = \mathrm{Spin}_8(2^n)$, $\mathrm{Spin}_9(2^n)$, or $\mathrm{SL}_5(2^n)$. If $G(n) = \mathrm{SU}_5(2^n)$, this is followed by the transpositions (I_1, I_2), (I_3, I_4), (I_6, I_7), (I_8, I_9), (I_{11}, I_{12}) and (I_{13}, I_{14}). We show in Chapter 6 that $H^1(G(n), M)$ is zero for most of the simple

$kG(n)$-modules $M = L(\lambda)$. To determine the remaining cohomology groups, we use the relationship between rational and generic cohomology via a bound on n (which, because of our reductions, will be independent of λ) discovered (for individual λ) by Cline et al. [7] and Andersen [3]. Thus, the main result of the paper can be stated as follows:

THEOREM *A) (D_4 version) If $n > 9$, then for simple $G(n)$ -modules M,*

$$H^1(G(n), M) \cong 2k$$

if M is a Frobenius twist of μ,

$$H^1(G(n), M) \cong k$$

if M is a Frobenius twist of Ψ, $\Psi\mu_1$, or $\mu\Theta_1^{abc}$ and is zero otherwise.
B) (A_4 version) If $n > 11$, then for simple $G(n)$ -modules M,

$$H^1(G(n), M) \cong k$$

if M or its dual is a Frobenius twist of Ψ, $\Delta\Lambda_1$, $\Lambda\Theta_1^*$, or $\Upsilon\Theta_1^*$ and is zero otherwise.
C) (B_4 version) If $n > 10$, then for simple $G(n)$ -modules M,

$$H^1(G(n), M) \cong k$$

if M is a Frobenius twist of $\Theta, \mu, \Psi, \Psi\mu_1$, or $\mu\sigma_1$, and is zero otherwise.
The same results hold for G (in all three cases) if $I_1, \ldots, I_{14}, R$ are allowed to be disjoint finite sets of nonnegative integers and conjugation is by $\mathbf{Z}$.

The result for G follows from the result for $G(n)$ because of Theorem 7.1 of Cline et al. [7], which asserts that the restriction map

$$\mathrm{Ext}^1_G(L(\mu), L(\nu)) \longrightarrow \mathrm{Ext}^1_{kG(n)}(L(\mu), L(\nu))$$

is injective if μ and ν are 2^n-restricted, and that it is an isomorphism if n is larger than a bound which depends on μ and ν.

CHAPTER 2

PRELIMINARY COMPUTATIONS

§2.1. Algebraic Group Computations

One of the techniques we shall use for analyzing the algebraic groups will be the exploitation of some intimate connections among the groups of types B_4, C_4, D_4 and F_4. It is well known that over an algebraically closed field k of characteristic 2 the groups of types B_l and C_l are isomorphic as abstract groups, but not as algebraic groups. More precisely, if G and $\tilde{G}$ are simply connected groups of types B_l and C_l, defined over $\mathbf{F}_2$, then there exist special isogenies [22]

$$\sigma : G \to \tilde{G}, \qquad \tau : \tilde{G} \to G$$

such that the composites $\tau \circ \sigma$ and $\sigma \circ \tau$ are the Frobenius morphisms of G and $\tilde{G}$. For any representation π of $\tilde{G}$, the composite $\pi \circ \sigma$ is a representation of G. If M is the module for π, we write $M^{(\sigma)}$ for the G-module thus obtained. Conversely, any G-module N on which G_σ acts trivially is of the form $M^{(\sigma)}$ for some $\tilde{G}$-module M, and we shall then write this module M as $N^{(\sigma^{-1})}$. Similar notation will be used for twisting and untwisting representations by τ. When dealing with the B_4 and C_4 groups, we shall therefore denote twisting by the n th power of the Frobenius map by $M^{(2^n)}$. The tangent maps $d\sigma$ and $d\tau$ between the Lie algebras have nonzero kernels $\mathfrak{g}_\sigma$ and $\tilde{\mathfrak{g}}_\tau$, which play an important role in the representation theory of the groups. They are the ideals generated by the short root spaces in the Lie algebras and Steinberg [21] used them to sharpen his tensor product theorem, reducing the

problem of determining the characters of all simple (rational) modules for G and $\tilde{G}$ to the case of simple modules for $\mathfrak{g}_\sigma$ or $\tilde{\mathfrak{g}}_\tau$. We observe that $\mathfrak{g}_\sigma$ is a quotient of the direct sum of l copies of $\mathfrak{sl}_2$ and the only G module on which it acts irreducibly is the 2^l-dimensional spin module. The algebra $\tilde{\mathfrak{g}}_\tau$ is isomorphic to a quotient of the Lie algebra of a simply connected group of type D_l, so its simple modules are a subset of the restricted simple modules for type D_l. Hence, Steinberg's observation implies that the characters of the simple modules for groups of types B_l, C_l and D_l will be known once they are known for D_l. In this way, the group of type D_4 becomes the natural starting point for our considerations.

For the most part, we adhere to the standard notational conventions from the literature on algebraic groups. If G is a semisimple, simply connected, algebraic group over an algebraically closed field of characteristic p, let T be a maximal torus of G and fix a choice of Borel subgroup containing T. Let $\mathbb{Z}\Phi$ denote the root lattice, Δ a (fixed) base of simple roots corresponding to the choice of Borel subgroup, $\mathbb{Z}^+\Delta$ the set of nonnegative integral linear combinations of positive roots, $X(T)$ the weight lattice, $X(T)^+$ the set of dominant weights, and let $X_1(T)$ be the set of p-restricted weights $= \{\nu \in X(T) : 0 \leq \langle \nu, \alpha_i^\vee \rangle < p \;\forall \alpha_i \in \Delta\}$. When dealing simultaneously with the algebraic groups of types B_4 and C_4, we adopt the notational convention of writing "$\tilde{x}$" for an object of type C_4 corresponding to an object "x" of type B_4.

The calculation of extensions of simple modules involves two main steps. The first consists of finding the G- and $\tilde{G}$-module structures of the groups $\mathrm{Ext}^1_{G_\sigma}(L(\lambda^0), L(\mu^0))$ and $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda'^0), \tilde{L}(\mu'^0))$, where $G_\sigma = \ker \sigma$, $\tilde{G}_\tau = \ker \tau$, and the modules $L(\lambda^0)$, $\tilde{L}(\mu')$, etc. are simple modules for G and $\tilde{G}$ which remain simple for G_σ and $\tilde{G}_\tau$. If we let D denote the subgroup of G generated by the long root subgroups (which is simply connected of type D_4), it turns out that the Frobenius kernel D_1 of D maps

onto $\bar{G}_\tau$ via σ and that the kernel is a subgroup of multiplicative type. Thus, we are able to use information about the D-module $\mathrm{Ext}^1_{D_1}(\bar{L}(\lambda^{(0)})^{(\sigma)}, \bar{L}(\mu^{(0)})^{(\sigma)})$ to help us in determining the $\bar{G}$ module structure of $\mathrm{Ext}^1_{\bar{G}_\tau}(\bar{L}(\lambda^{(0)}), \bar{L}(\mu^{(0)}))$. If E denotes one of the Ext^1 groups above, then the second step is to compute the socle of the G-module $E \otimes L(\lambda)$ for all dominant weights λ (and similarly for $\bar{G}$). In the groups of rank 4, it is only necessary to consider those λ which are 2-restricted. Once these socles are determined, the extensions are computed via certain isomorphisms obtained from the Lyndon-Hochschild-Serre spectral sequences for the pairs (G, G_σ) and $(\bar{G}, \bar{G}_\tau)$. They can then essentially be read off from a reasonably small set of tables. The Ext^1 groups for types A_4 and D_4 are computed similarly using their Frobenius kernels (and the appropriate spectral sequences.)

We need a small amount of information about extensions of simple modules in order to get started. For example, we will want to know some Weyl module structure in order to compute the socles of tensor products of restricted simple modules. A fundamental property of Weyl modules is that if $\lambda \not> \mu$, then $\dim_k(\mathrm{Ext}_G(L(\nu), L(\lambda)))$ is equal to the multiplicity of $L(\lambda)$ in the second radical layer of $V(\mu)$. We shall frequently compute "good" filtrations of certain modules in order to obtain information about their socles. A filtration G-module is said to be *good* if the subquotients are isomorphic to induced modules $H^0(\lambda)$ for various $\lambda \in X_+$. There is also the dual notion of a Weyl filtration. The important facts for our purposes about a finite-dimensional G-module M with a good filtration are the following:

(1) The multiplicity of $H^0(\lambda)$ as a subquotient is $\dim_k \mathrm{Hom}_G(V(\lambda), M)$.

(2) If $H^0(\lambda)$ and $H^0(\mu)$ are both good filtration factors and $\lambda \not> \mu$, then M has a good filtration in which the factor $H^0(\mu)$ appears above the factor $H^0(\lambda)$.

(3) If the module M' also has a good filtration, then so does $M \otimes M'$.

A proof of (1) can be found in Jantzen [12, II,4.16], and (2) follows from a standard property of Weyl modules [12, II, 2.14]. The deeper fact (3) is proved by Donkin [10, 7.3.1] with a few exceptions and in general by Mathieu [14]. Given a module with a good filtration, the multiplicities of the subquotients can be determined from the weight multiplicities in the module. Another important concept related to module extensions is that of "linkage". The action of the "affine Weyl group" on the set of weights is generated by the "dot" action of the Weyl group (where $w \cdot v = w(v+\rho)-\rho$), together with translations by p-multiples of roots. Two weights are said to be "linked" if they are in the same orbit under the affine Weyl group. From Jantzen's Sum Formula [12], it is clear that $L(\lambda)$ and $L(\mu)$ cannot appear as compositions of the same Weyl module unless λ and μ are linked. For future reference, we record some tables of composition factors of certain Weyl modules which will be used throughout the paper. These can easily be computed using Jantzen's Sum Formula, together with Freudenthal's Formula (or by utilizing the tables of weight multiplicities of Bremner et al. [5]). For convenience, we do not repeat information which can clearly be obtained from that already recorded by applying a graph automorphism or by symmetry via relabelling.

TABLE 1-4 WEYL MODULE COMPOSITION FACTORS (A_4)

Weyl module	radical (composition factors)
$V(\lambda_1)$	$\emptyset$
$V(\lambda_2)$	$\emptyset$
$V(\lambda_1+\lambda_2)$	$\emptyset$
$V(\lambda_1+\lambda_4)$	$\emptyset$
$V(\lambda_2+\lambda_4)$	$L(\lambda_1)$
$V(\lambda_2+\lambda_3)$	k
$V(\lambda_1+\lambda_2+\lambda_4)$	$L(\lambda_2), L(2\lambda_1)$
$V(\lambda_1+\lambda_2+\lambda_3)$	$\emptyset$
$V(2\lambda_1)$	$L(\lambda_2)$
$V(2\lambda_3)$	$L(\lambda_2+\lambda_4)$
$V(3\lambda_4)$	$L(\lambda_2)$
$V(2\lambda_1+\lambda_4)$	$L(\lambda_1), L(\lambda_2+\lambda_4)$
$V(\lambda_2+2\lambda_4)$	$2k, L(\lambda_2+\lambda_3)$
$V(3\lambda_1+\lambda_4)$	$L(\lambda_3+\lambda_4)$
$V(\lambda_1+2\lambda_3)$	$L(\lambda_1+\lambda_2+\lambda_4)$
$V(2\lambda_2+\lambda_3)$	$L(2\lambda_1), L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_1+2\lambda_3)$
$V(\lambda_1+\lambda_2+2\lambda_4)$	$L(\lambda_1+\lambda_2+\lambda_3)$
$V(2\lambda_2+3\lambda_4)$	$L(\lambda_2+\lambda_4), L(2\lambda_3), 2L(2\lambda_1+\lambda_4), L(3\lambda_2), L(\lambda_1+\lambda_3+3\lambda_4)$
$V(3\lambda_1+2\lambda_2+\lambda_4)$	$2L(2\lambda_2+\lambda_3+\lambda_4), L(2\lambda_1+\lambda_2+\lambda_3+\lambda_4), L(4\lambda_1+\lambda_3+\lambda_4)$
$V(3\lambda_2+2\lambda_3)$	$2L(\lambda_2), 2L(2\lambda_1), 2L(3\lambda_4), 2L(2\lambda_2+2\lambda_4)$, $L(\lambda_3+2\lambda_3+\lambda_4), 2L(2\lambda_1+2\lambda_3+\lambda_4), L(4\lambda_3)$, $L(\lambda_1+2\lambda_2+\lambda_3+\lambda_4), L(4\lambda_2+\lambda_4)$

Table 1-5 Weyl module composition factors (D_4)

Weyl module	radical (composition factors)
$V(\delta_1)$	$\emptyset$
$V(\delta_2)$	$2k$
$V(\delta_1+\delta_2)$	$\emptyset$
$V(\delta_2+\delta_4)$	$L(\delta_1)$
$V(\delta_1+\delta_3+\delta_4)$	$2k, 3L(\delta_2), L(2\delta_1), L(2\delta_3), L(2\delta_4)$
$V(\delta_2+\delta_3+\delta_4)$	$L(\delta_1), L(\delta_3+\delta_4)$
$V(3\delta_1)$	$L(\delta_3+\delta_4)$
$V(\delta_1+2\delta_3)$	$L(\delta_1+\delta_2)$
$V(2\delta_1+\delta_3+\delta_4)$	$2L(\delta_1), L(\delta_3+\delta_4), L(3\delta_1), L(\delta_2+\delta_3+\delta_4)$
$V(3\delta_1+\delta_4)$	$L(\delta_1+\delta_2), L(\delta_1+2\delta_3), L(\delta_1+2\delta_4)$
$V(\delta_1+2\delta_2)$	$3L(\delta_1), L(\delta_2+\delta_3+\delta_4), L(2\delta_1+\delta_3+\delta_4)$
$V(\delta_1+\delta_2+2\delta_3)$	$4L(\delta_1), L(\delta_3+\delta_4), L(3\delta_1),$ $L(\delta_2+\delta_3+\delta_4), L(2\delta_1+\delta_3+\delta_4), L(\delta_1+2\delta_2)$
$V(5\delta_1)$	$2L(\delta_1), L(2\delta_1+\delta_3+\delta_4), L(\delta_1+2\delta_2)$
$V(3\delta_1+2\delta_4)$	$L(\delta_1+\delta_2), 2L(\delta_1+2\delta_3), L(\delta_1+2\delta_4),$ $L(\delta_3+3\delta_4), L(3\delta_1+\delta_2)$
$V(\delta_1+2\delta_3+2\delta_4)$	$5L(\delta_1), L(\delta_3+\delta_4), 2L(3\delta_1),$ $L(\delta_2+\delta_3+\delta_4), L(2\delta_1+\delta_3+\delta_4), L(\delta_1+2\delta_2),$ $L(\delta_1+\delta_2+2\delta_3), L(\delta_1+\delta_2+2\delta_4)$
$V(2\delta_2+\delta_3+\delta_4)$	$6L(\delta_1), 3L(\delta_3+\delta_4), 3L(3\delta_1), L(\delta_2+\delta_3+\delta_4),$ $L(2\delta_1+\delta_3+\delta_4), 2L(\delta_1+2\delta_2), L(\delta_1+\delta_2+2\delta_3),$ $L(\delta_1+\delta_2+2\delta_4), L(\delta_1+2\delta_3+2\delta_4)$
$V(2\delta_1+\delta_2+\delta_3+\delta_4)$	$7L(\delta_1), 4L(\delta_2+\delta_4), 4L(3\delta_1), L(\delta_2+\delta_3+\delta_4),$ $2L(2\delta_1+\delta_3+\delta_4), 2L(\delta_1+2\delta_2), L(\delta_1+\delta_2+2\delta_3),$ $L(\delta_1+\delta_2+2\delta_4), L(\delta_1+2\delta_3+2\delta_4), L(2\delta_2+\delta_3+\delta_4)$
$V(3\delta_1+2\delta_2)$	$4L(\delta_3+\delta_4), 2L(3\delta_1), 2L(2\delta_2+\delta_3+\delta_4),$ $L(2\delta_1+\delta_2+\delta_3+\delta_4), L(4\delta_1+\delta_3+\delta_4)$

TABLE 1-6 WEYL MODULE COMPOSITION FACTORS (C_4)

Weyl module	radical (composition factors)
$\bar{V}(\bar{\omega}_1)$	$\emptyset$
$\bar{V}(\bar{\omega}_2)$	k
$\bar{V}(\bar{\omega}_3)$	$\emptyset$
$\bar{V}(2\bar{\omega}_1)$	$2k, \bar{L}(\bar{\omega}_2)$

§2.2. Character Multiplication Tables

This section is included mainly for reference purposes. We need to obtain information about the structure of tensor products of many different pairs of simple modules over the algebraic groups as well as over the finite groups. For example, when we compute the extensions over the algebraic groups we will mainly need information about the socles of various tensor products (in fact, much of our work will be in reducing the computation of the extensions for the modules over the algebraic group to questions about the socle structure of certain tensor products, which can then be listed conveniently in table form). When we deal with the finite groups, we are more interested in the totality of isomorphism types of composition factors which appear within each tensor product. In many instances we need to show that there are no nonzero homomorphisms from a given module with simple head into a given tensor product of simple modules; sometimes all that is required is to show that the tensor product has no composition factors isomorphic to the head of the given module.

LEMMA 2.2.1 *The composition factors which appear in each tensor product of pairs of restricted simple modules are as indicated in Tables 2-1 to 2-9. (We have omitted those tensor products in B_4 which are immediately obtainable from given ones, e.g.,* $\Lambda\sigma \otimes \Psi \cong \sigma \otimes (\Lambda \otimes \Psi)$.*)*

(REMARK In many cases, we are only interested in which modules appear as composition factors in a product of restricted modules, and not in their multiplicities as composition factors.)

PROOF The weight multiplicities of the restricted simple modules can be completely determined from those of the Weyl modules [6] using the Jantzen Sum Formula [12]. The tensor products are then computed by calculating the weight orbits under the Weyl group and then multiplying the appropriate formal characters. ∎

TABLE 2-1 (D_4 TENSOR PRODUCTS)

product	composition factors
$\Theta^a \otimes \Theta^a$	$4k, 2\mu, \Theta_1^a$
$\Theta^a \otimes \Theta^b$	$2\Theta^c, \Lambda^c$
$\Theta^a \otimes \mu$	Λ^a, Δ^a
$\Theta^a \otimes \Lambda^a$	$2k, 4\mu, 2\Theta_1^{bc}, \Psi$
$\Theta^a \otimes \Lambda^b$	$2\Delta^c, \Theta^c\Theta_1^a$
$\Theta^a \otimes \Delta^a$	$10k, 8\mu, 2\Theta_1^{bc}, 4\Theta_1^a, 3\Psi, 2\mu_1, \mu\Theta_1^a$
$\Theta^a \otimes \Delta^b$	$2\Theta^c, 2\Lambda^c, 2\Delta^c, \Theta^c\Theta_1^b, \Gamma^c$
$\Theta^a \otimes \Psi$	$2\Theta^a, 2\Gamma^a, \Lambda^a\Theta_1^a$
$\Theta^a \otimes \Gamma^a$	$8k, 4\mu, 4\Theta_1^{abc}, 4\Psi, 4\mu_1, 2\mu\Theta_1^{bc}, \Lambda_1^a, S$
$\Theta^a \otimes \Gamma^b$	$8\Theta^c, 2\Lambda^c, 2\Theta^c\Theta_1^c, 4\Gamma^c, 3\Lambda^c\Theta_1^c, 2\Theta^c\mu_1, \Delta^c\Theta_1^a$

TABLE 2-1 (CONTINUED)

product	composition factors
$\mu \otimes \mu$	$8k, 4\mu, 2\Theta_1^{abc}, 2\Psi, \mu_1$
$\mu \otimes \Lambda^a$	$2\Theta^a, 2\Delta^a, \Theta^a\Theta_1^{bc}, \Gamma^a$
$\mu \otimes \Delta^a$	$8\Theta^a, 4\Lambda^a, 2\Delta^a, \Theta^a\Theta_1^{bc}, 2\Theta^a\Theta_1^a, 3\Gamma^a, 2\Lambda^a\Theta_1^a, \Theta^a\mu_1$
$\mu \otimes \Psi$	$12k, 6\mu, 6\Theta_1^{abc}, 2\Psi, 4\mu_1, 2\mu\Theta_1^{abc}, \Lambda_1^{abc}, S$
$\mu \otimes \Gamma^a$	$24\Theta^a, 8\Lambda^a, 2\Delta^a, 2\Theta^a\Theta_1^{bc}, 8\Theta^a\Theta_1^a, 6\Gamma^a,$ $\Lambda^a\Theta_1^{bc}, 6\Lambda^a\Theta_1^a, 8\Theta^a\mu_1, 3\Delta^a\Theta_1^{bc}, 2\Theta^a\Lambda_1^a, \Lambda^a\mu_1$
$\Lambda^a \otimes \Lambda^a$	$16k, 8\mu, 4\Theta_1^{bc}, 6\Theta_1^a, 4\Psi, 4\mu_1, 2\mu\Theta_1^{bc}, \Lambda_1^a$
$\Lambda^a \otimes \Lambda^b$	$4\Theta^c, 4\Lambda^c, 2\Theta^c\Theta_1^c, 2\Gamma^c, \Lambda^c\Theta_1^c$
$\Lambda^a \otimes \Delta^a$	$20k, 16\mu, 10\Theta_1^{bc}, 8\Theta_1^a, 6\Psi, 4\mu_1, 2\mu\Theta_1^{abc}, \Lambda_1^{bc}, S$
$\Lambda^a \otimes \Delta^b$	$12\Theta^c, 4\Lambda^c, 4\Delta^c, 2\Theta^c\Theta_1^{abc}, 4\Gamma^c, \Lambda^c\Theta_1^b, 3\Lambda^c\Theta_1^c, 2\Theta^c\mu_1, \Delta^c\Theta_1^b$
$\Lambda^a \otimes \Psi$	$16\Theta^a, 6\Lambda^a, 6\Theta^a\Theta_1^a, 4\Gamma^a, 4\Lambda^a\Theta_1^a, 4\Theta^a\mu_1, 2\Delta^a\Theta_1^{bc}, \Theta^a\Lambda_1^a$
$\Lambda^a \otimes \Gamma^a$	$68k, 32\mu, 34\Theta_1^{bc}, 32\Theta_1^a, 10\Psi, 24\mu_1, 12\mu\Theta_1^{bc},$ $10\mu\Theta_1^a, 8\Lambda_1^{abc}, 2\Theta_2^{bc}, 2S, 4\mu\mu_1, 3\Psi\Theta_1^{bc}, 2\Delta_1^{bc}, \mu\Lambda_1^a$
$\Lambda^a \otimes \Gamma^b$	$32\Theta^c, 12\Lambda^c, 4\Delta^c, 4\Theta^c\Theta_1^{ab}, 12\Theta^c\Theta_1^a, 8\Gamma^c, 8\Lambda^c\Theta_1^c,$ $2\Lambda^c\Theta_1^a, 8\Theta^a\mu_1, 2\Delta^c\Theta_1^c, 4\Delta^c\Theta_1^{ab}, \Theta^c\Lambda_1^b, 3\Theta^a\Lambda_1^c, 2\Lambda^c\mu_1, \Gamma^c\Theta_1^c$
$\Delta^a \otimes \Delta^a$	$72k, 44\mu, 28\Theta_1^{bc}, 32\Theta_1^a, 16\Psi, 20\mu_1, 6\mu\Theta_1^{bc},$ $12\mu\Theta_1^a, 4\Lambda_1^{abc}, 2\Theta_2^a, 2S, 2\mu\mu_1, 2\Psi\Theta_1^a, \Delta_1^a$
$\Delta^a \otimes \Delta^b$	$32\Theta^c, 14\Lambda^c, 8\Delta^c, 4\Theta^c\Theta_1^{ab}, 10\Theta^c\Theta_1^c, 10\Gamma^c,$ $\Lambda^c\Theta_1^{ab}, 7\Lambda^c\Theta_1^c, 6\Theta^c\mu_1, 3\Delta^c\Theta_1^{ab}, 2\Theta^c\Lambda_1^c, \Lambda^c\mu_1$
$\Delta^a \otimes \Psi$	$32\Theta^a, 12\Lambda^a, 8\Delta^a, 8\Theta^a\Theta_1^{bc}, 12\Theta^a\Theta_1^a, 8\Gamma^a, 2\Lambda^a\Theta_1^{bc},$ $8\Lambda^a\Theta_1^a, 8\Theta^a\mu_1, 4\Delta^a\Theta_1^{bc}, 2\Delta^a\Theta_1^a, \Theta^a\Lambda_1^{bc}, 3\Theta^a\Lambda_1^a, 2\Lambda^a\mu_1, \Gamma^a\Theta_1^a$
$\Delta^a \otimes \Gamma^a$	$212k, 88\mu, 102\Theta_1^{bc}, 100\Theta_1^a, 28\Psi, 86\mu_1, 38\mu\Theta_1^{bc}, 32\mu\Theta_1^a, 26\Lambda_1^{bc}, 28\Lambda_1^a,$ $10\Theta_2^{bc}, 6\Theta_2^a, 4S, 12\mu\mu_1, 7\Psi\Theta_1^{bc}, 8\Psi\Theta_1^a, 8\Delta_1^{abc}, 3\mu\Lambda_1^{abc}, 2\Psi_1, \Psi\mu_1$
$\Delta^a \otimes \Gamma^b$	$72\Theta^c, 28\Lambda^c, 16\Delta^c, 20\Theta^c\Theta_1^a, 16\Theta^c\Theta_1^b, 28\Theta^c\Theta_1^c, 18\Gamma^c, 8\Lambda^c\Theta_1^a,$ $8\Lambda^c\Theta_1^b, 18\Lambda^c\Theta_1^c, 18\Theta^c\mu_1, 10\Delta^c\Theta_1^a, 8\Delta^c\Theta_1^b, 6\Delta^c\Theta_1^c, 4\Theta^c\Lambda_1^{ab},$ $6\Theta^c\Lambda_1^c, \Theta^a\Theta_2^a, 4\Lambda^c\mu_1, 3\Gamma^c\Theta_1^a, 2\Gamma^c\Theta_1^c, 2\Delta^c\mu_1, 2\Lambda^c\Lambda_1^b, \Theta^c\Delta_1^c$
$\Psi \otimes \Psi$	$118k, 48\mu, 56\Theta_1^{abc}, 12\Psi, 48\mu_1, 20\mu\Theta_1^{abc},$ $18\Lambda_1^{abc}, 6\Theta_2^{abc}, 8\mu\mu_1, 4\Psi\Theta_1^{abc}, 4\Delta_1^{abc}, 2\mu\Lambda_1^{abc}, \Psi_1$

TABLE 2-1 (CONTINUED)

product	composition factors
$\Psi \otimes \Gamma^a$	$56\Theta^a, 24\Lambda^a, 24\Delta^a, 32\Theta^a\Theta_1^{bc}, 24\Theta^a\Theta_1^a, 10\Gamma^a, 14\Lambda^a\Theta_1^{bc}, 12\Lambda^a\Theta_1^a, 16\Theta^a\mu_1, 8\Delta^a\Theta_1^{bc}, 10\Delta^a\Theta_1^a, 8\Theta^a\Lambda_1^{bc}, 8\Theta^a\Lambda_1^a, 2\Theta^a\Theta_2^{bc}, 4\Lambda^a\mu_1, 4\Gamma^a\Theta_1^{bc}, 2\Gamma^a\Theta_1^a, 3\Lambda^a\Lambda_1^{bc}, \Lambda^a\Lambda_1^a, 4\Delta^a\mu_1, 2\Theta^a\Delta_1^{bc}, \Delta^a\Lambda_1^a$
$\Gamma^a \otimes \Gamma^a$	$648k, 248\mu, 296\Theta_1^{bc}, 298\Theta_1^a, 64\Psi, 304\mu_1, 108\mu\Theta_1^{bc}, 104\mu\Theta_1^a, 92\Lambda_1^{bc}, 100\Lambda_1^a, 52\Theta_2^{bc}, 40\Theta_2^a, 4S, 56\mu\mu_1, 24\Psi\Theta_1^{bc}, 20\Psi\Theta_1^a, 36\Delta_1^{bc}, 32\Delta_1^a, 6\mu\Theta_2^{bc}, 14\mu\Lambda_1^{bc}, 20\mu\Lambda_1^a, 8\Theta_1^a\Theta_2^{bc}, 4\Theta_1^b\Theta_2^c, 4\Theta_1^c\Theta_2^b, 18\Psi_1, 4\mu_2, 8\Psi\mu_1, 2S\Theta_1^{bc}, 2\mu\Delta_1^{bc}, 2\Psi\Lambda_1^a, \Gamma_1^a$
$\Gamma^a \otimes \Gamma^b$	$172\Theta^c, 80\Lambda^c, 48\Delta^c, 64\Theta^c\Theta_1^{ab}, 60\Theta^c\Theta_1^c, 28\Gamma^c, 28\Lambda^c\Theta_1^{ab}, 40\Lambda^c\Theta_1^c, 64\Theta^c\mu_1, 24\Delta^c\Theta_1^{ab}, 20\Delta^c\Theta_1^c, 16\Theta^a\Lambda_1^{ab}, 20\Theta^c\Lambda_1^c, 8\Theta^c\Theta_2^{ab}, 10\Theta^c\Theta_2^c, 22\Lambda^c\mu_1, 8\Gamma^c\Theta_1^{ab}, 12\Gamma^c\Theta_1^c, 6\Lambda^c\Lambda_1^{ab}, 4\Lambda^c\Lambda_1^c, 7\Lambda^c\Theta_2^c, 8\Delta^c\mu_1, 8\Theta^c\Delta^{ab}, 8\Theta^c\Delta^c, 3\Delta^c\Lambda^{ab}, 2\Delta^c\Lambda_1^c, 2\Theta^c\Psi_1, 2\Gamma^c\mu_1, \Lambda^c\Delta_1^c$
$S \otimes \Theta^a$	$32\Theta^a, 12\Lambda^a, 12\Theta^a\Theta_1^a, 8\Gamma^a, 8\Lambda^a\Theta_1^a, 8\Theta^a\mu_1, 4\Delta^a\Theta_1^{bc}, 3\Theta^a\Lambda_1^a, 2\Lambda^a\mu_1, \Gamma^a\Theta_1^a$
$S \otimes \mu$	$176k, 72\mu, 64\Theta_1^{abc}, 22\Psi, 74\mu_1, 30\mu\Theta_1^{abc}, 24\Lambda_1^{abc}, 8\Theta_2^{abc}, 2S, 12\mu\mu_1, 8\Psi\Theta_1^{abc}, 8\Delta_1^{abc}, 3\mu\Lambda_1^{abc}, 2\Psi_1, \Psi\mu_1$
$S \otimes \Lambda^a$	$64\Theta^a, 24\Lambda^a, 24\Delta^a, 32\Theta^a\Theta_1^{bc}, 24\Theta^a\Theta_1^a, 16\Gamma^a, 14\Lambda^a\Theta_1^{bc}, 18\Lambda^a\Theta_1^a, 18\Theta^a\mu_1, 8\Delta^a\Theta_1^{bc}, 10\Delta^a\Theta_1^a, 8\Theta^a\Lambda_1^{bc}, 6\Theta^a\Lambda_1^a, 4\Lambda^a\mu_1, 4\Gamma^a\Theta_1^{bc}, 2\Gamma^a\Theta_1^a, 3\Lambda^a\Lambda_1^{bc}, 4\Delta^a\mu_1, 2\Theta^a\Delta_1^{bc}, \Delta^a\Lambda_1^a$
$S \otimes \Delta^a$	$220\Theta^a, 96\Lambda^a, 48\Delta^a, 64\Theta^a\Theta_1^{bc}, 96\Theta^a\Theta_1^a, 44\Gamma^a, 28\Lambda^a\Theta_1^{bc}, 52\Lambda^a\Theta_1^a, 78\Theta^a\mu_1, 30\Delta^a\Theta_1^{bc}, 20\Delta^a\Theta_1^a, 16\Theta^a\Lambda_1^{bc}, 24\Theta^a\Lambda_1^a, 8\Theta^a\Theta_2^{bc}, 24\Lambda^a\mu_1, 8\Gamma^a\Theta_1^{bc}, 12\Gamma^a\Theta_1^a, 6\Lambda^a\Lambda_1^{bc}, 4\Lambda^a\Lambda_1^a, 8\Lambda^a\Theta_2^a, 8\Delta^a\mu_1, 4\Theta^a\Delta_1^{bc}, 6\Theta^a\Delta_1^a, 3\Delta^a\Lambda_1^{bc}, 2\Delta^a\Lambda_1^a, 2\Theta^a\Psi_1, 2\Gamma^a\mu_1, \Lambda^a\Delta_1^a$
$S \otimes \Psi$	$832k, 298\mu, 378\Theta_1^{abc}, 72\Psi, 440\mu_1, 136\mu\Theta_1^{abc}, 134\Lambda_1^{abc}, 60\Theta_2^{abc}, 6S, 88\mu\mu_1, 30\Psi\Theta_1^{abc}, 64\Delta_1^{abc}, 10\mu\Theta_2^{abc}, 28\mu\Lambda_1^{abc}, 8\Theta_1^a\Theta_2^{bc}, 8\Theta_1^b\Theta_2^{ac}, 8\Theta_1^c\Theta_2^{ab}, 32\Psi_1, 12\Psi\mu_1, 2S\Theta_1^{abc}, 8\mu_2, 3\Psi\Lambda_1^{abc}, 4\mu\Delta_1^{abc}, 2\Gamma_1^{abc}, \mu\Psi_1$
$S \otimes \Gamma^a$	$552\Theta^a, 236\Lambda^a, 160\Delta^a, 240\Theta^a\Theta_1^{bc}, 242\Theta^a\Theta_1^a, 88\Gamma^a, 112\Lambda^a\Theta_1^{bc}, 120\Lambda^a\Theta_1^a, 248\Theta^a\mu_1, 78\Delta^a\Theta_1^{bc}, 72\Delta^a\Theta_1^a, 72\Theta^a\Lambda_1^{bc}, 76\Theta^a\Lambda_1^a, 40\Theta^a\Theta_2^{abc}, 84\Lambda^a\mu_1, 36\Gamma^a\Theta_1^{bc}, 32\Gamma^a\Theta_1^a, 32\Lambda^a\Lambda_1^{bc}, 20\Lambda^a\Lambda_1^a, 8\Lambda^a\Theta_2^{bc}, 20\Lambda^a\Theta_2^a, 48\Delta^a\mu_1, 36\Theta^a\Delta_1^{bc}, 32\Theta^a\Delta_1^a, 8\Delta^a\Theta_2^{bc}, 14\Delta^a\Lambda_1^{bc}, 20\Delta^a\Lambda_1^a, 6\Theta^a\Theta_1^b\Theta_2^c, 6\Theta^a\Theta_1^c\Theta_2^b, 4\Theta^a\Theta_1^a\Theta_2^{bc}, 16\Theta^a\Psi_1, 12\Gamma^a\mu_1, 4\Lambda^a\Delta_1^{bc}, 8\Lambda^a\Delta_1^a, 4\Theta^a\mu_2, 2\Gamma^a\Lambda_1^{bc}, 3\Gamma^a\Lambda_1^a, 2\Lambda^a\Psi_1, 2\Delta^a\Delta_1^{bc}, \Theta^a\Gamma_1^a$

TABLE 2-1 (CONTINUED)

product	composition factors
$S \otimes S$	$8248k, 2616\mu, 3456\Theta_1^{abc}, 596\Psi, 5152\mu_1, 1212\mu\Theta_1^{abc}, 1488\Lambda_1^{abc}, 1152\Theta_2^{abc}, 48S, 1120\mu\mu_1, 272\Psi\Theta_1^{abc}, 960\Delta_1^{abc}, 184\mu\Theta_2^{abc}, 352\mu\Lambda_1^{abc}, 172\Theta_1^a\Theta_2^{bc}, 172\Theta_1^b\Theta_2^{ac}, 172\Theta_1^c\Theta_2^{ab}, 40\Theta_{\{1,2\}}^{abc}, 636\Psi_1, 176\Psi\mu_1, 20S\Theta_1^{abc}, 208\mu_2, 56\Psi\Lambda_1^{abc}, 120\mu\Delta_1^{abc}, 20\Psi\Theta_2^{abc}, 92\Gamma_1^{abc}, 32\mu_1\Theta_2^{abc}, 14\mu\Theta_1^a\Theta_2^{bc}, 14\mu\Theta_1^b\Theta_2^{ac}, 14\mu\Theta_1^c\Theta_2^{ab}, 52\mu\Psi_1, 16(\Lambda_1\Theta_2)^{abc}, 4\Lambda_2^{abc}, 8S\mu_1, 8\mu\mu_2, 8\Psi\Delta_1^{abc}, 4\Theta_1^{abc}\mu_2, 2S\Lambda_1^{abc}, 2\Psi\Psi_1, 2\mu\Gamma_1^{abc}, S_1$

TABLE 2-2 (A_4 TENSOR PRODUCTS)

product	composition factors
$\Theta \otimes \Theta$	$2\Lambda, \Theta_1$
$\Theta \otimes \Theta^*$	k, μ
$\Theta \otimes \Lambda$	Λ^*, Σ^*
$\Theta \otimes \Lambda^*$	$2\Theta^*, \Delta^*$
$\Theta \otimes \mu$	$3\Theta, 2\Delta, \Theta^*\Theta_1$
$\Theta \otimes \Delta$	Σ, Υ
$\Theta \otimes \Delta^*$	$2k, 2\Psi, \Lambda^*\Theta_1$
$\Theta \otimes \Sigma$	$3\Lambda^*, 2\Theta_1^*, \Upsilon^*$
$\Theta \otimes \Sigma^*$	$2\Theta^*, 3\Delta^*, 2\Lambda_1, \Lambda\Theta_1$
$\Theta \otimes \Psi$	$2\Delta, \Lambda_1^*, \Gamma$
$\Theta \otimes \Upsilon$	$\Lambda^*, 2\Theta_1^*, 3\Upsilon^*, 2\Theta^*\Lambda_1, \Delta\Theta_1$
$\Theta \otimes \Upsilon^*$	$\Delta^*, 2\Gamma^*, \Sigma\Theta_1$
$\Theta \otimes \Gamma$	$2\Lambda, 4\Theta_1, 4T, 3\Theta\Lambda_1^*, 2\Lambda^*\Lambda_1, \Psi\Theta_1$
$\Theta \otimes \Gamma^*$	$4k\ 3\Psi, 2\Lambda\Theta_1^*, \Theta^*\Lambda_1^*, S$

TABLE 2-2 (CONTINUED)

product	composition factors
$\Lambda\otimes\Lambda$	$2\Theta^*, 2\Delta^*, \Lambda_1$
$\Lambda\otimes\Lambda^*$	$2k, \mu, \Psi$
$\Lambda\otimes\mu$	$3\Lambda, 2\Theta_1, \Sigma, \Upsilon$
$\Lambda\otimes\Delta$	$2\Lambda^*, 2\Theta_1^*, 2\Upsilon^*, \Theta^*\Lambda_1$
$\Lambda\otimes\Delta^*$	$\Theta, 2\Delta, \Lambda_1^*, \Theta^*\Theta_1, \Gamma$
$\Lambda\otimes\Sigma$	$3\Theta^*, 2\Delta^*, \Theta\Theta_1^*, \Gamma^*$
$\Lambda\otimes\Sigma^*$	$4k, \mu, 3\Psi, 2\Lambda^*\Theta_1, \Theta\Lambda_1$
$\Lambda\otimes\Psi$	$\Lambda, 2\Theta_1, \Sigma, 3\Upsilon, 2\Theta\Lambda_1^*, \Lambda^*\Lambda_1$
$\Lambda\otimes\Upsilon$	$\Theta^*, 2\Delta^*, \Lambda_1, \Theta\Theta_1^*, 3\Gamma^*, 2\Sigma\Theta_1, \mu\Lambda_1$
$\Lambda\otimes\Upsilon^*$	$6k, 4\Psi, 2\Lambda\Theta_1^*, 2\Lambda^*\Theta_1, \Theta^*\Lambda_1^*, \mu_1, S$
$\Lambda\otimes\Gamma$	$3\Lambda^*, 6\Theta_1^*, \Sigma^*, 2\Theta\Theta_1, 6\Upsilon^*, 6\Theta^*\Lambda_1, 3\Lambda\Lambda_1^*, 3\Delta\Theta_1, 2\Delta_1^*, \Delta^*\Lambda_1$
$\Lambda\otimes\Gamma^*$	$2\Theta, 3\Delta, 2\Lambda_1^*, \Lambda^*\Theta_1^*, 4\Gamma, 3\Sigma^*\Theta_1^*, 2\mu\Lambda_1^*, \Sigma\Lambda_1$
$\mu\otimes\mu$	$8k, 2\mu, 4\Psi, 2\Lambda\Theta_1^*, 2\Lambda^*\Theta_1, \mu_1$
$\mu\otimes\Delta$	$2\Theta, 3\Delta, 2\Lambda_1^*, \Lambda^*\Theta_1^*, 2\Gamma, \Sigma^*\Theta_1^*$
$\mu\otimes\Sigma$	$4\Lambda, 2\Theta_1, 2\Sigma, 2\Theta^*\Theta_1^*, 3\Upsilon, 2\Theta\Lambda_1^*, \Delta^*\Theta_1^*$
$\mu\otimes\Psi$	$8k, 6\Psi, 2\Lambda\Theta_1^*, 2\Lambda^*\Theta_1, \Theta^*\Lambda_1^*, \Theta\Lambda_1, S$
$\mu\otimes\Upsilon$	$5\Lambda, 12\Theta_1, \Sigma, 2\Theta^*\Theta_1^*, 8\Upsilon, 6\Theta\Lambda_1^*, 4\Lambda^*\Lambda_1, 3\Delta^*\Theta_1^*, 2\Delta_1, 2\Psi\Theta_1, \Lambda\mu_1$
$\mu\otimes\Gamma$	$4\Theta, 6\Delta, 4\Lambda_1^*, 4\Theta^*\Theta_1, 7\Gamma, 2\Lambda\Lambda_1,$ $2\Delta^*\Theta_1, 4\Sigma^*\Theta_1^*, \Sigma_1^*, 3\mu\Lambda_1^*, 2\Sigma\Lambda_1, \Gamma^*\Theta_1$
$\Delta\otimes\Delta$	$4\Lambda, 6\Theta_1, 2\Theta^*\Theta_1^*, 4\Upsilon, 4\Theta\Lambda_1^*, 2\Lambda^*\Lambda_1, 2\Delta^*\Theta_1^*, \Delta_1$
$\Delta\otimes\Delta^*$	$6k, \mu, 3\Psi, 2\Lambda\Theta_1^*, 2\Lambda^*\Theta_1, \Theta^*\Lambda_1^*, \Theta\Lambda_1, \mu_1, S$
$\Delta\otimes\Sigma$	$2\Lambda^*, 4\Theta_1^*, 2\Sigma^*, 4\Upsilon^*, \mu\Theta_1^*, 3\Theta^*\Lambda_1, 2\Lambda\Lambda_1^*, \Psi\Theta_1^*$

TABLE 2-2 (CONTINUED)

product	composition factors
$\Delta\otimes\Sigma^*$	$3\Theta^*, 2\Delta^*, \Theta\Theta_1^*, 3\Gamma^*, 2\Sigma\Theta_1, \mu\Lambda_1$
$\Delta\otimes\Psi$	$4\Theta, 3\Delta, 2\Lambda_1^*, 2\Theta^*\Theta_1, \Lambda^*\Theta_1^*, 4\Gamma, \Lambda\Lambda_1, 3\Sigma^*\Theta_1^*, 2\mu\Lambda_1^*, \Sigma\Lambda_1$
$\Delta\otimes\Upsilon$	$8\Lambda^*, 18\Theta_1^*, 2\Sigma^*, 6\Theta\Theta_1, 8\Upsilon^*, \mu\Theta_1^*, 9\Theta^*\Lambda_1,$ $8\Lambda\Lambda_1^*, 4\Delta\Theta_1, 3\Psi\Theta_1^*, 4\Delta_1^*, 2\Delta^*\Lambda_1, 2\Lambda^*\mu_1, \Theta\Delta_1$
$\Delta\otimes\Upsilon^*$	$6\Theta^*, 6\Delta^*, 5\Lambda_1, 4\Theta\Theta_1^*, 2\Lambda\Theta_1, 5\Gamma^*, 2\Lambda^*\Lambda_1^*,$ $4\Sigma\Theta_1, 2\Delta\Theta_1^*, 3\mu\Lambda_1, \Theta_1^*\mu_1, \Sigma_1, 2\Sigma^*\Lambda_1^*, \Gamma\Theta_1^*$
$\Delta\otimes\Gamma$	$11\Lambda, 26\Theta_1, 3\Sigma, 8\Theta^*\Theta_1^*, 11\Upsilon, 12\Theta\Lambda_1^*, 2\mu\Theta_1, 8\Lambda^*\Lambda_1,$ $6\Delta^*\Theta_1^*, 6\Delta_1, 4\Psi\Theta_1, \Sigma^*\Lambda_1, 3\Delta\Lambda_1^*, 3\Lambda\mu_1, 2\Theta^*\Delta_1^*, \Upsilon^*\Lambda_1$
$\Delta\otimes\Gamma^*$	$26k, 2\mu, 12\Psi, 9\Lambda\Theta_1^*, 10\Lambda^*\Theta_1, 8\Theta^*\Lambda_1^*, 8\mu_1, 6\Theta\Lambda_1,$ $\Sigma\Theta_1^*, 2S, 4\Delta^*\Lambda_1^*, 3\Delta\Lambda_1, 3\Upsilon\Theta_1^*, 2\Psi_1, 2\Theta\Sigma_1, \Lambda^*\Delta_1$
$\Sigma\otimes\Sigma$	$4\Theta^*, 6\Delta^*, 4\Lambda_1, 4\Theta\Theta_1^*, 2\Gamma^*, 2\Lambda^*\Lambda_1^*, 2\Delta\Theta_1^*, \Sigma_1$
$\Sigma\otimes\Sigma^*$	$8k, 2\mu, 4\Psi, 2\Lambda\Theta_1^*, 2\Lambda^*\Theta_1, \mu_1, S$
$\Sigma\otimes\Psi$	$8\Lambda, 10\Theta_1, 2\Theta^*\Theta_1^*, 7\Upsilon, 8\Theta\Lambda_1^*, 3\Lambda^*\Lambda_1, 3\Delta^*\Theta_1^*, 2\Delta_1, \Delta\Lambda_1^*$
$\Sigma\otimes\Upsilon$	$6\Theta^*, 6\Delta^*, 4\Lambda_1, 3\Theta\Theta_1^*, 2\Lambda\Theta_1, 8\Gamma^*,$ $4\Sigma\Theta_1, 2\Delta\Theta_1^*, 3\mu\Lambda_1, \Theta_1^*\mu_1, 2\Sigma^*\Lambda_1^*, \Gamma\Theta_1^*$
$\Sigma\otimes\Upsilon^*$	$16k, \mu, 9\Psi, 6\Lambda\Theta_1^*, 6\Lambda^*\Theta_1, 3\Theta^*\Lambda_1^*, 4\mu_1, 4\Theta\Lambda_1, 2S, 2\Delta^*\Lambda_1^*, 2\Upsilon\Theta_1^*, \Theta\Sigma_1$
$\Sigma\otimes\Gamma$	$12\Lambda^*, 26\Theta_1^*, 2\Sigma^*, 10\Theta\Theta_1, 13\Upsilon^*, 14\Theta^*\Lambda_1, 8\Lambda\Lambda_1^*,$ $7\Delta\Theta_1, 4\Psi\Theta_1^*, 6\Delta_1^*, 3\Delta^*\Lambda_1, 3\Lambda^*\mu_1, 2\Theta\Delta_1, \Upsilon\Lambda_1^*$
$\Sigma\otimes\Gamma^*$	$8\Theta, 9\Delta, 12\Lambda_1^*, 8\Theta^*\Theta_1, 4\Lambda^*\Theta_1^*, 8\Gamma, 2\Theta_2^*, 4\Lambda\Lambda_1,$ $8\Sigma^*\Theta_1^*, 4\mu\Lambda_1^*, 3\Upsilon^*\Theta_1^*, 2\Sigma\Lambda_1, 2\Theta^*\Delta_1, 2\Psi\Lambda_1^*, \Lambda\Sigma_1$
$\Psi\otimes\Psi$	$18k, 2\mu, 8\Psi, 8\Lambda\Theta_1^*, 8\Lambda^*\Theta_1, 4\Theta^*\Lambda_1^*, 4\Theta\Lambda_1, 4\mu_1, 2S, 2\Delta^*\Lambda_1^*, 2\Delta\Lambda_1, \Psi_1$
$\Psi\otimes\Upsilon$	$13\Lambda, 28\Theta_1, 3\Sigma, 10\Theta^*\Theta_1^*, 12\Upsilon, 14\Theta\Lambda_1^*, 2\mu\Theta_1, 8\Lambda^*\Lambda_1,$ $7\Delta^*\Theta_1^*, 6\Delta_1, 4\Psi\Theta_1, \Sigma^*\Lambda_1, 3\Delta\Lambda_1^*, 3\Lambda\mu_1, 2\Theta^*\Delta_1^*, \Upsilon^*\Lambda_1$
$\Psi\otimes\Gamma$	$18\Theta, 14\Delta, 16\Lambda_1^*, 10\Theta^*\Theta_1, 8\Lambda^*\Theta_1^*, 10\Gamma, 8\Lambda\Lambda_1, 8\Delta^*\Theta_1,$ $8\Sigma^*\Theta_1^*, 4\Sigma_1^*, 4\Theta\mu_1, 8\mu\Lambda_1^*, 4\Sigma\Lambda_1, 2\Gamma^*\Theta_1, 3\Psi\Lambda_1^*, 2\Lambda^*\Delta_1^*, 2\Upsilon\Lambda_1, \Theta\Psi_1$
$\Upsilon\otimes\Upsilon$	$16\Theta^*, 16\Delta^*, 28\Lambda_1, 18\Theta\Theta_1^*, 12\Lambda\Theta_1, 6\Theta_2, 10\Gamma^*,$ $8\Lambda^*\Lambda_1^*, 8\Sigma\Theta_1, 8\Delta\Theta_1^*, 8\mu\Lambda_1, 4\Theta^*\mu_1, 4\Sigma_1, 4\Sigma^*\Lambda_1^*,$ $4\Upsilon\Theta_1, 4\Psi\Lambda_1, 4\Theta\Delta_1^*, 2\Gamma\Theta_1^*, 2\Lambda\Delta_1, 2\Delta^*\mu_1, 2\Lambda^*\Sigma_1^*, \Upsilon_1$
$\Upsilon\otimes\Upsilon^*$	$46k, 5\mu, 18\Psi, 18\Lambda\Theta_1^*, 18\Lambda^*\Theta_1, 13\Theta^*\Lambda_1^*, 13\Theta\Lambda_1, 17\mu_1,$ $2\Sigma\Theta_1^*, 2\Sigma^*\Theta_1, 3S, 6\Delta^*\Lambda_1^*, 6\Delta\Lambda_1, 4\Upsilon^*\Theta_1, 4\Upsilon\Theta_1^*, 4\Psi_1,$ $\mu\mu_1, 3\Theta\Sigma_1, 3\Theta^*\Sigma_1^*, 2\Lambda\Delta_1^*, 2\Lambda^*\Delta_1, \Psi\mu_1$

Table 2-2 (continued)

product	composition factors
$\Upsilon \otimes \Gamma$	$25\Lambda^{*}, 58\Theta_1^{*}, 9\Sigma^{*}, 22\Theta\Theta_1, 23\Upsilon^{*}, 8\mu\Theta_1^{*}, 30\Theta^{*}\Lambda_1, 17\Lambda\Lambda_1^{*}, 16\Delta\Theta_1, 8\Psi\Theta_1^{*}, 16\Delta_1^{*}, 2\Theta^{*}\Theta_2, 4\Sigma\Lambda_1^{*}, 9\Delta^{*}\Lambda_1, 8\Lambda^{*}\mu_1, 4\Gamma\Theta_1, 2\Lambda_2, 4\Theta\Delta_1, 3\Sigma^{*}\mu_1, 2\Upsilon\Lambda_1^{*}, \Lambda\Sigma_1^{*}, 2\mu\Delta_1^{*}, 2\Gamma^{*}\Lambda_1, \Sigma\Sigma_1^{*}$
$\Upsilon \otimes \Gamma^{*}$	$26\Theta, 22\Delta, 44\Lambda_1^{*}, 26\Theta^{*}\Theta_1, 19\Lambda^{*}\Theta_1^{*}, 14\Gamma, 10\Theta_2^{*}, 12\Lambda\Lambda_1, 10\Delta^{*}\Theta_1, 11\Sigma^{*}\Theta_1^{*}, 8\Sigma_1^{*}, 8\Theta\mu_1, 8\mu\Lambda_1^{*}, 7\Upsilon^{*}\Theta_1^{*}, 5\Sigma\Lambda_1, 2\Gamma^{*}\Theta_1, 6\Theta^{*}\Delta_1, 6\Psi\Lambda_1^{*}, 3\Delta\mu_1, 4\Lambda^{*}\Delta_1^{*}, 3\Upsilon\Lambda_1, 3\Lambda\Sigma_1, 2\Upsilon_1^{*}, 2\Theta\Psi_1, \Delta^{*}\Delta_1$
$\Gamma \otimes \Gamma$	$42\Lambda, 108\Theta_1, 8\Sigma, 40\Theta^{*}\Theta_1^{*}, 32\Upsilon, 40\Theta\Lambda_1^{*}, 6\mu\Theta_1, 36\Lambda^{*}\Lambda_1, 20\Delta^{*}\Theta_1^{*}, 32\Delta_1, 20\Psi\Theta_1, 8\Lambda^{*}\Theta_2, 4\Sigma^{*}\Lambda_1, 14\Delta\Lambda_1^{*}, 14\Lambda\mu_1, 16\Theta^{*}\Delta_1^{*}, 4\Theta_1^{*}\Theta_2, 4\Lambda_2^{*}, 4\Theta\Sigma_1^{*}, 8\Upsilon^{*}\Lambda_1, 2\Gamma\Lambda_1^{*}, 4\Theta^{*}\Lambda_2, 2S\Theta_1, 2\Delta\Sigma_1^{*}, 2\Delta^{*}\Delta_1^{*}, 2\Lambda\Psi_1, \Gamma_1$
$\Gamma \otimes \Gamma^{*}$	$88k, 12\mu, 36\Psi, 34\Lambda\Theta_1^{*}, 34\Lambda^{*}\Theta_1, 24\Theta^{*}\Lambda_1^{*}, 24\Theta\Lambda_1, 33\mu_1, 8\Sigma\Theta_1^{*}, 8\Sigma^{*}\Theta_1, 6S, 12\Delta^{*}\Lambda_1^{*}, 12\Delta\Lambda_1, 8\Upsilon\Theta_1^{*}, 8\Upsilon^{*}\Theta_1, 8\Psi_1, 4\mu\mu_1, 6\Theta\Sigma_1, 6\Theta^{*}\Sigma_1^{*}, 4\Lambda^{*}\Delta_1, 4\Lambda\Delta_1^{*}, 3\Gamma^{*}\Lambda_1^{*}, 3\Gamma\Lambda_1, 2\Psi\mu_1, 2\Sigma\Delta_1^{*}, 2\Sigma^{*}\Delta_1, \mu\Psi_1$
$S \otimes \Theta$	$5\Gamma, 4\Sigma^{*}\Theta_1^{*}, 3\mu\Lambda_1^{*}, 2\Sigma\Lambda_1, \Gamma^{*}\Theta_1$
$S \otimes \Lambda$	$8\Lambda, 24\Theta_1, 8\Theta^{*}\Theta_1^{*}, 10\Upsilon, 12\Theta\Lambda_1^{*}, 8\Lambda^{*}\Lambda_1, 8\Delta^{*}\Theta_1^{*}, 8\Delta_1, 4\Psi\Theta_1, 3\Delta\Lambda_1^{*}, 3\Lambda\mu_1, 2\Theta^{*}\Delta_1^{*}, \Upsilon^{*}\Lambda_1$
$S \otimes \mu$	$40k, 18\Psi, 16\Lambda\Theta_1^{*}, 16\Lambda^{*}\Theta_1, 12\Theta^{*}\Lambda_1^{*}, 12\Theta\Lambda_1, 16\mu_1, 4S, 6\Delta^{*}\Lambda_1^{*}, 6\Delta\Lambda_1, 4\Upsilon\Theta_1^{*}, 4\Upsilon^{*}\Theta_1, 4\Psi_1, 3\Theta\Sigma_1, 3\Theta^{*}\Sigma_1^{*}, 2\Lambda^{*}\Delta_1, 2\Lambda\Delta_1^{*}, \Psi\mu_1$
$S \otimes \Delta$	$20\Theta, 18\Delta, 40\Lambda_1^{*}, 24\Theta^{*}\Theta_1, 16\Lambda^{*}\Theta_1^{*}, 10\Gamma, 8\Theta_2^{*}, 12\Lambda\Lambda_1, 10\Delta^{*}\Theta_1, 8\Sigma^{*}\Theta_1^{*}, 8\Sigma_1^{*}, 8\Theta\mu_1, 6\mu\Lambda_1^{*}, 6\Upsilon^{*}\Theta_1^{*}, 4\Sigma\Lambda_1, 2\Gamma^{*}\Theta_1, 8\Theta^{*}\Delta_1, 6\Psi\Lambda_1^{*}, 3\Delta\mu_1, 4\Lambda^{*}\Delta_1^{*}, 3\Upsilon\Lambda_1, 3\Lambda\Sigma_1, 2\Upsilon_1^{*}, 2\Theta\Psi_1, \Delta^{*}\Delta_1$
$S \otimes \Sigma$	$16\Lambda, 48\Theta_1, 8\Sigma, 16\Theta^{*}\Theta_1^{*}, 20\Upsilon, 24\Theta\Lambda_1^{*}, 8\mu\Theta_1, 16\Lambda^{*}\Lambda_1, 12\Delta^{*}\Theta_1^{*}, 12\Delta_1, 8\Psi\Theta_1, 4\Sigma^{*}\Lambda_1, 6\Delta\Lambda_1^{*}, 6\Lambda\mu_1, 4\Theta^{*}\Delta_1^{*}, 4\Gamma^{*}\Theta_1^{*}, 2\Upsilon^{*}\Lambda_1, 3\Sigma\mu_1, 2\mu\Delta_1, 2\Gamma\Lambda_1^{*}, \Sigma^{*}\Sigma_1$
$S \otimes \Psi$	$80k, 19\mu, 32\Psi, 32\Lambda\Theta_1^{*}, 32\Lambda^{*}\Theta_1, 24\Theta^{*}\Lambda_1^{*}, 24\Theta\Lambda_1, 32\mu_1, 8\Sigma\Theta_1^{*}, 8\Sigma^{*}\Theta_1, 4S, 12\Delta^{*}\Lambda_1^{*}, 12\Delta\Lambda_1, 8\Upsilon\Theta_1^{*}, 8\Upsilon^{*}\Theta_1, 8\Psi_1, 4\mu\mu_1, 8\Theta\Sigma_1, 8\Theta^{*}\Sigma_1^{*}, 4\Lambda^{*}\Delta_1, 4\Lambda\Delta_1^{*}, 3\Gamma^{*}\Lambda_1^{*}, 3\Gamma\Lambda_1, 2\Psi\mu_1, 2\Sigma\Delta_1^{*}, 2\Sigma^{*}\Delta_1, \mu\Psi_1$
$S \otimes \Upsilon$	$56\Lambda, 180\Theta_1, 16\Sigma, 60\Theta^{*}\Theta_1^{*}, 38\Upsilon, 64\Theta\Lambda_1^{*}, 16\mu\Theta_1, 51\Lambda^{*}\Lambda_1, 34\Delta^{*}\Theta_1^{*}, 58\Delta_1, 8\Theta\Theta_2^{*}, 28\Psi\Theta_1, 10\Lambda^{*}\Theta_2, 8\Sigma^{*}\Lambda_1, 21\Delta\Lambda_1^{*}, 22\Lambda\mu_1, 24\Theta^{*}\Delta_1^{*}, 8\Theta_1^{*}\Theta_2, 8\Lambda_2^{*}, 6\Gamma^{*}\Theta_1^{*}, 8\Theta\Sigma_1^{*}, 11\Upsilon^{*}\Lambda_1, 4\Lambda^{*}\Sigma_1, 6\Sigma\mu_1, 4\mu\Delta_1, 4\Gamma\Lambda_1^{*}, 6\Theta^{*}\Lambda_2, 2S\Theta_1, 2\Sigma^{*}\Sigma_1, 6\Delta\Sigma_1^{*}, 3\Upsilon\mu_1, 4\Delta^{*}\Delta_1^{*}, 4\Lambda\Psi_1, 2\Theta\Upsilon_1^{*}, 2\Gamma_1, 2\Psi\Delta_1, \Lambda^{*}\Upsilon_1$
$S \otimes \Gamma$	$88\Theta, 56\Delta, 160\Lambda_1^{*}, 92\Theta^{*}\Theta_1, 58\Lambda^{*}\Theta_1^{*}, 32\Gamma, 40\Theta_2^{*}, 52\Lambda\Lambda_1, 44\Delta^{*}\Theta_1, 32\Sigma^{*}\Theta_1^{*}, 48\Sigma_1^{*}, 37\Theta\mu_1, 24\mu\Lambda_1^{*}, 20\Upsilon^{*}\Theta_1^{*}, 28\Sigma\Lambda_1, 8\Lambda\Theta_2, 18\Gamma^{*}\Theta_1, 32\Theta^{*}\Delta_1, 24\Psi\Lambda_1^{*}, 14\Delta\mu_1, 26\Lambda^{*}\Delta_1^{*}, 18\Upsilon\Lambda_1, 14\Lambda\Sigma_1, 8\Sigma\Theta_2, 4\Theta^{*}\Theta_1^{*}\Theta_2, 16\Upsilon_1^{*}, 8\mu\Sigma_1^{*}, 4\Theta^{*}\Lambda_2^{*}, 16\Theta\Psi_1, 8\Lambda^{*}\Lambda_2, 4\Sigma^{*}\Delta_1^{*}, 8\Delta^{*}\Delta_1, 4\Theta_1^{*}\Lambda_2, 3\Psi\Sigma_1^{*}, 2\Gamma\mu_1, 2S\Lambda_1^{*}, 2\Upsilon^{*}\Delta_1^{*}, 2\Lambda\Upsilon_1, 2\Delta\Psi_1, \Theta^{*}\Gamma_1$

TABLE 2-2 (CONTINUED)

product	composition factors
$S \otimes S$	$544k, 64\mu, 120\Psi, 218\Lambda\Theta_1^*, 218\Lambda^*\Theta_1, 184\Theta^*\Lambda_1^*, 184\Theta\Lambda_1, 272\mu_1, 56\Sigma\Theta_1^*, 56\Sigma^*\Theta_1, 40\Theta\Theta_2, 40\Theta^*\Theta_2^*, 16S, 94\Delta^*\Lambda_1^*, 94\Delta\Lambda_1, 64\Upsilon\Theta_1^*, 64\Upsilon^*\Theta_1, 152\Psi_1, 28\mu\mu_1, 72\Theta\Sigma_1, 72\Theta^*\Sigma_1^*, 20\Delta\Theta_2, 20\Delta^*\Theta_2^*, 78\Lambda^*\Delta_1, 78\Lambda\Delta_1^*, 24\Gamma^*\Lambda_1^*, 24\Gamma\Lambda_1, 36\Psi\mu_1, 32\Lambda_1\Theta_2^*, 32\Lambda_1^*\Theta_2, 14\Delta\Sigma_1, 14\Delta^*\Sigma_1^*, 24\Sigma\Delta_1^*, 24\Sigma^*\Delta_1, 8\Lambda\Lambda_2, 8\Lambda^*\Lambda_2^*, 14\Lambda\Theta_1\Theta_2^*, 14\Lambda^*\Theta_1^*\Theta_2, 4\Theta_1\Lambda_2, 4\Theta_1^*\Lambda_2^*, 16\Theta\Upsilon_1, 16\Theta^*\Upsilon_1^*, 4\mu_2, 16\mu\Psi_1, 8\Upsilon\Delta_1^*, 8\Upsilon^*\Delta_1, 4\Sigma\Lambda_2, 4\Sigma^*\Lambda_2^*, 4\Theta\Theta_1\Lambda_2^*, 4\Theta^*\Theta_1^*\Lambda_2, 4\Psi\Psi_1, 2S\mu_1, 2\Gamma\Sigma_1, 2\Gamma^*\Sigma_1^*, 2\Delta\Upsilon_1, 2\Delta^*\Upsilon_1^*, 2\Lambda\Gamma_1^*, 2\Lambda^*\Gamma_1, S_1$

TABLE 2-3 (B_4 TENSOR PRODUCTS)

product	composition factors
$\Theta \otimes \Theta$	$4k, 2\mu, \Theta_1$
$\Theta \otimes \mu$	Λ, Δ
$\Theta \otimes \Lambda$	$2k, 4\mu, 2\sigma_1, \Psi$
$\Theta \otimes \Delta$	$10k, 8\mu, 4\Theta_1, 2\sigma_1, 3\Psi, 2\mu_1, \mu\Theta_1$
$\Theta \otimes \Psi$	$2\Theta, 2\Gamma, \Lambda\Theta_1$
$\Theta \otimes \Gamma$	$8k, 4\mu, 4\Theta_1, 4\sigma_1, 4\Psi, 4\mu_1, 2\mu\sigma_1, \Lambda_1, \Sigma$
$\Theta \otimes \Sigma$	$32\Theta, 12\Lambda, 12\Theta\Theta_1, 8\Gamma, 8\Lambda\Theta_1, 8\Theta\mu_1, 4\Delta\sigma_1, 3\Theta\Lambda_1, 2\Lambda\mu_1, \Gamma\Theta_1$
$\mu \otimes \mu$	$6k, 4\mu, 2\Theta_1, 2\sigma_1, 2\Psi, \mu_1$
$\mu \otimes \Lambda$	$2\Theta, 2\Delta, \Theta\sigma_1, \Gamma$
$\mu \otimes \Delta$	$8\Theta, 4\Lambda, 2\Delta, 2\Theta\Theta_1, \Theta\sigma_1, 3\Gamma, 2\Lambda\Theta_1, \Theta\mu_1$
$\mu \otimes \Psi$	$12k, 6\mu, 6\Theta_1, 4\sigma_1, 2\Psi, 4\mu_1, 2\mu\Theta_1, 2\mu\sigma_1, (\Theta\sigma)_1, \Lambda_1, \Sigma$
$\mu \otimes \Gamma$	$24\Theta, 8\Lambda, 2\Delta, 8\Theta\Theta_1, 2\Theta\sigma_1, 8\Gamma, 6\Lambda\Theta_1, 6\Theta\mu_1, \Lambda\sigma_1, 3\Delta\sigma_1, 2\Theta\Lambda_1, \Lambda\mu_1$
$\mu \otimes \Sigma$	$176k, 72\mu, 84\Theta_1, 48\sigma_1, 22\Psi, 74\mu_1, 30\mu\Theta_1, 8\Theta_2, 24\mu\sigma_1, 18(\Theta\sigma)_1, 24\Lambda_1, 2\Sigma, 8\sigma_2, 6\Psi\Theta_1, 12\mu\mu_1, 8\Delta_1, 6\Psi\sigma_1, 8(\mu\sigma)_1, 3\mu\Lambda_1, 3\mu(\Theta\sigma)_1, 2\Psi_1, \Psi\mu_1$
$\Lambda \otimes \Lambda$	$16k, 8\mu, 8\Theta_1, 4\sigma_1, 4\Psi, 4\mu_1, 2\mu\sigma_1, \Lambda_1$

Table 2-3 (continued)

product	composition factors
$\Lambda\otimes\Delta$	$20k, 16\mu, 8\Theta_1, 8\sigma_1, 8\Psi, 4\mu_1, 2\mu\Theta_1, 2\mu\sigma_1, (\Theta\sigma)_1, \Sigma$
$\Lambda\otimes\Psi$	$12\Theta, 8\Lambda, 6\Theta\Theta_1, 4\Gamma, 4\Lambda\Theta_1, 4\Theta\mu_1, 2\Delta\sigma_1, \Theta\Lambda_1$
$\Lambda\otimes\Gamma$	$68k, 32\mu, 32\Theta_1, 22\sigma_1, 10\Psi, 24\mu_1, 10\mu\Theta_1, 12\mu\sigma_1,$ $8(\Theta\sigma)_1, 8\Lambda_1, 2\Sigma, 2\sigma_2, 4\mu\mu_1, 3\Psi\sigma_1, 2(\mu\sigma)_1, \mu\Lambda_1$
$\Lambda\otimes\Sigma$	$64\Theta, 24\Lambda, 24\Delta, 24\Theta\Theta_1, 20\Theta\sigma_1, 18\Gamma, 18\Lambda\Theta_1,$ $16\Theta\mu_1, 10\Delta\Theta_1, 8\Lambda\sigma_1, 8\Delta\sigma_1, 6\Theta\Lambda_1, 6\Theta(\Theta\sigma)_1, 4\Lambda\mu_1,$ $2\Gamma\Theta_1, 4\Delta\mu_1, 4\Gamma\sigma_1, 3\Lambda(\Theta\sigma)_1, 2\Theta(\mu\sigma)_1, \Delta\Lambda_1$
$\Delta\otimes\Delta$	$72k, 44\mu, 32\Theta_1, 20\sigma_1, 18\Psi, 20\mu_1, 12\mu\Theta_1,$ $2\Theta_2, 6\mu\sigma_1, 4(\Theta\sigma)_1, 4\Lambda_1, 2\Sigma, 2\Psi\Theta_1, 2\mu\mu_1, \Delta_1$
$\Delta\otimes\Psi$	$32\Theta, 12\Lambda, 6\Delta, 12\Theta\Theta_1, 4\Theta\sigma_1, 8\Gamma, 8\Lambda\Theta_1, 8\Theta\mu_1,$ $2\Delta\Theta_1, 2\Lambda\sigma_1, 4\Delta\sigma_1, 3\Theta\Lambda_1, \Theta(\Theta\sigma)_1, 2\Lambda\mu_1, \Gamma\Theta_1$
$\Delta\otimes\Gamma$	$212k, 88\mu, 100\Theta_1, 82\sigma_1, 28\Psi, 88\mu_1, 32\mu\Theta_1,$ $8\Theta_2, 32\mu\sigma_1, 20(\Theta\sigma)_1, 28\Lambda_1, 4\Sigma, 10\sigma_2, 6\Psi\Theta_1,$ $12\mu\mu_1, 6\Delta_1, 7\Psi\sigma_1, 6(\mu\sigma)_1, 3\mu\Lambda_1, 3\mu(\Theta\sigma)_1, 2\Psi_1, \Psi\mu_1$
$\Delta\otimes\Sigma$	$220\Theta, 96\Lambda, 48\Delta, 96\Theta\Theta_1, 40\Theta\sigma_1, 44\Gamma, 52\Lambda\Theta_1,$ $78\Theta\mu_1, 20\Delta\Theta_1, 16\Lambda\sigma_1, 8\Theta\Theta_2, 24\Delta\sigma_1, 24\Theta\Lambda_1, 12\Theta(\Theta\sigma)_1,$ $24\Lambda\mu_1, 8\Theta\sigma_2, 12\Gamma\Theta_1, 8\Delta\mu_1, 6\Lambda\Theta_2, 6\Theta\Delta_1, 8\Gamma\sigma_1,$ $8\Lambda(\Theta\sigma)_1, 4\Lambda\Lambda_1, 4\Theta(\mu\sigma)_1, 2\Delta\Lambda_1, 3\Delta(\Theta\sigma)_1, 2\Gamma\mu_1, 2\Theta\Psi_1, \Lambda\Delta_1$
$\Psi\otimes\Psi$	$118k, 48\mu, 56\Theta_1, 32\sigma_1, 12\Psi, 48\mu_1, 20\mu\Theta_1, 6\Theta_2, 18\mu\sigma_1, 12(\Theta\sigma)_1,$ $18\Lambda_1, 6\sigma_2, 4\Psi\Theta_1, 8\mu\mu_1, 4\Delta_1, 4\Psi\sigma_1, 4(\mu\sigma)_1, 2\mu\Lambda_1, 2\mu(\Theta\sigma)_1, \Psi_1$
$\Psi\otimes\Gamma$	$56\Theta, 24\Lambda, 24\Delta, 24\Theta\Theta_1, 20\Theta\sigma_1, 10\Gamma, 12\Lambda\Theta_1, 18\Theta\mu_1,$ $10\Delta\Theta_1, 8\Lambda\sigma_1, 8\Delta\sigma_1, 6\Theta\Lambda_1, 6\Theta(\Theta\sigma)_1, 4\Lambda\mu_1, 2\Theta\sigma_2,$ $2\Gamma\Theta_1, 4\Delta\mu_1, 4\Gamma\sigma_1, 3\Lambda(\Theta\sigma)_1, \Lambda\Lambda_1, 2\Theta(\mu\sigma)_1, \Delta\Lambda_1$
$\Psi\otimes\Sigma$	$332k, 256\mu, 378\Theta_1, 200\sigma_1, 72\Psi, 440\mu_1, 136\mu\Theta_1, 80\Theta_2, 88\mu\sigma_1,$ $86(\Theta\sigma)_1, 134\Lambda_1, 6\Sigma, 60\sigma_2, 30\Psi\Theta_1, 88\mu\mu_1, 64\Delta_1, 24\Psi\sigma_1, 10\mu\Theta_2,$ $44(\mu\sigma)_1, 28\mu\Lambda_1, 24\mu(\Theta\sigma)_1, 8\sigma_1\Theta_2, 32\Psi_1, 12\Psi\mu_1, 10\mu\sigma_2, 8\Theta_1\sigma_2, 2\Sigma\Theta_1,$ $8\mu_2, 8(\Lambda\sigma)_1, 4\mu\Delta_1, 2\Sigma\sigma_1, 4\mu(\mu\sigma)_1, 3\Psi\Lambda_1, 3\Psi(\Theta\sigma)_1, 2\Gamma_1, 2(\Delta\sigma)_1, \mu\Psi_1$
$\Gamma\otimes\Gamma$	$648k, 248\mu, 298\Theta_1, 168\sigma_1, 64\Psi, 304\mu_1, 104\mu\Theta_1, 40\Theta_2,$ $84\mu\sigma_1, 64(\Theta\sigma)_1, 100\Lambda_1, 4\Sigma, 52\sigma_2, 20\Psi\Theta_1, 56\mu\mu_1, 32\Delta_1,$ $24\Psi\sigma_1, 28(\mu\sigma)_1, 20\mu\Lambda_1, 12\mu(\Theta\sigma)_1, 18\Psi_1, 8\Psi\mu_1, 8\mu\sigma_2,$ $4\Theta_1\sigma_2, 4\mu_2, 4(\Lambda\sigma)_1, 2\Sigma\sigma_1, 2\mu(\mu\sigma)_1, 2\Psi\Lambda_1, \Gamma_1$
$\Gamma\otimes\Sigma$	$552\Theta, 236\Lambda, 160\Delta, 242\Theta\Theta_1, 144\Theta\sigma_1, 88\Gamma, 120\Lambda\Theta_1, 248\Theta\mu_1,$ $72\Delta\Theta_1, 56\Lambda\sigma_1, 40\Theta\Theta_2, 52\Delta\sigma_1, 76\Theta\Lambda_1, 48\Theta(\Theta\sigma)_1, 84\Lambda\mu_1, 40\Theta\sigma_2,$ $32\Gamma\Theta_1, 48\Delta\mu_1, 20\Lambda\Theta_2, 32\Theta\Delta_1, 32\Gamma\sigma_1, 28\Lambda(\Theta\sigma)_1, 20\Lambda\Lambda_1, 24\Theta(\mu\sigma)_1,$ $20\Delta\Lambda_1, 12\Delta(\Theta\sigma)_1, 8\Lambda\sigma_2, 12\Gamma\mu_1, 16\Theta\Psi_1, 8\Lambda\Delta_1, 8\Delta\sigma_2, 4\Theta\mu_2, 4\Theta\Theta_1\sigma_2,$ $8\Theta(\Lambda\sigma)_1, 4\Lambda(\mu\sigma)_1, 2\Gamma(\Theta\sigma)_1, 3\Gamma\Lambda_1, 2\Lambda\Psi_1, 2\Delta(\mu\sigma)_1, \Theta\Gamma_1$

TABLE 2-3 (CONTINUED)

product	composition factors
$\Sigma\otimes\Sigma$	$8248k, 2618\mu, 3456\Theta_1, 1720\sigma_1, 596\Psi, 5152\mu_1, 1212\mu\Theta_1, 1152\Theta_2, 688\mu\sigma_1, 792(\Theta\sigma)_1, 1488\Lambda_1, 48\Sigma, 1144\sigma_2, 272\Psi\Theta_1, 1120\mu\mu_1, 960\Delta_1, 176\Psi\sigma_1, 184\mu\Theta_2, 576(\mu\sigma)_1, 40\Theta_1\Theta_2, 352\mu\Lambda_1, 260\mu(\Theta\sigma)_1, 112\sigma_1\Theta_2, 636\Psi_1, 176\Psi\mu_1, 184\mu\sigma_2, 172\Theta_1\sigma_2, 20\Sigma\Theta_1, 208\mu_2, 132(\Lambda\sigma)_1, 120\mu\Delta_1, 20\Psi\Theta_2, 16\Sigma\sigma_1, 40\sigma_1\sigma_2, 32\mu_1\Theta_2, 88\mu(\mu\sigma)_1, 56\Psi\Lambda_1, 48\Psi(\Theta\sigma)_1, 92\Gamma_1, 60(\Delta\sigma)_1, 12\mu\sigma_1\Theta_2, 52\mu\Psi_1, 20\Psi\sigma_2, 16\Lambda_1\Theta_2, 8\Sigma\mu_1, 32\mu_1\sigma_2, 14\mu(\Lambda\sigma)_1, 8\mu\mu_2, 14\mu\Theta_1\sigma_2, 8\Psi\Delta_1, 4\Theta_1\mu_2, 4(\Theta\sigma)_2, 4\Lambda_2, 18(\Psi\sigma)_1, 8\Psi(\mu\sigma)_1, 4\sigma_1\mu_2, 2\Sigma(\Theta\sigma)_1, 2\Sigma\Lambda_1, 2\Psi\Psi_1, 2\mu(\Delta\sigma)_1, 2\mu\Gamma_1, \Sigma_1$
$\sigma\otimes\sigma$	$8k, 4\Theta, 4\mu, 2\Lambda, \sigma_1$
$\sigma\otimes\Theta\sigma$	$20k, 8\Theta, 16\mu, 4\Theta_1, 4\Lambda, 4\sigma_1, 4\Delta, 2\Psi, \Theta\sigma_1$
$\sigma\otimes\mu\sigma$	$24k, 4\Theta, 24\mu, 8\Theta_1, 4\Lambda, 8\sigma_1, 8\Delta, 8\Psi, 4\mu_1, 2\Theta\sigma_1, 2\Gamma, \mu\sigma_1$
$\sigma\otimes\Lambda\sigma$	$40k, 8\Theta, 32\mu, 12\Theta_1, 8\Lambda, 16\sigma_1, 8\Delta, 12\Psi, 8\mu_1, 4\Theta\sigma_1, 4\Gamma, 4\mu\sigma_1, 2\Lambda_1, \Lambda\sigma_1$
$\sigma\otimes\Delta\sigma$	$80k, 32\Theta, 64\mu, 32\Theta_1, 18\Lambda, 24\sigma_1, 18\Delta, 8\Theta\Theta_1, 24\Psi, 16\mu_1, 4\Theta\sigma_1, 8\mu\Theta_1, 12\Gamma, 4\mu\sigma_1, 8\Lambda\Theta_1, 4\Theta\mu_1, 2(\Theta\sigma)_1, 2\Sigma, \Delta\sigma_1$
$\sigma\otimes\Psi\sigma$	$48k, 40\Theta, 24\mu, 24\Theta_1, 12\Lambda, 16\sigma_1, 12\Theta\Theta_1, 18\Psi, 16\mu_1, 8\mu\Theta_1, 18\Gamma, 8\mu\sigma_1, 12\Lambda\Theta_1, 8\Theta\mu_1, 4(\Theta\sigma)_1, 4\Lambda_1, 4\Sigma, 4\Delta\sigma_1, 2\Theta\Lambda_1, \Psi\sigma_1$
$\sigma\otimes\Gamma\sigma$	$168k, 96\Theta, 80\mu, 80\Theta_1, 32\Lambda, 60\sigma_1, 8\Delta, 32\Theta\Theta_1, 36\Psi, 64\mu_1, 8\Theta\sigma_1, 20\mu\Theta_1, 32\Gamma, 32\mu\sigma_1, 24\Lambda\Theta_1, 24\Theta\mu_1, 12(\Theta\sigma)_1, 20\Lambda_1, 4\Lambda\sigma_1, 8\Sigma, 4\sigma_2, 8\mu\mu_1, 12\Delta\sigma_1, 8\Theta\Lambda_1, 6\Psi\sigma_1, 4\Lambda\mu_1, 4(\mu\sigma)_1, 2\mu\Lambda_1, \Gamma\sigma_1$
$\sigma\otimes S$	$704k, 256\Theta, 288\mu, 336\Theta_1, 96\Lambda, 192\sigma_1, 48\Delta, 96\Theta\Theta_1, 88\Psi, 296\mu_1, 40\Theta\sigma_1, 120\mu\Theta_1, 32\Theta_2, 64\Gamma, 96\mu\sigma_1, 64\Lambda\Theta_1, 64\Theta\mu_1, 72(\Theta\sigma)_1, 96\Lambda_1, 20\Delta\Theta_1, 16\Lambda\sigma_1, 16\Sigma, 32\sigma_2, 24\Psi\Theta_1, 48\mu\mu_1, 32\Delta\sigma_1, 24\Delta_1, 24\Theta\Lambda_1, 12\Theta(\Theta\sigma)_1, 24\Psi\sigma_1, 16\Lambda\mu_1, 24(\mu\sigma)_1, 8\Gamma\Theta_1, 8\Delta\mu_1, 12\mu\Lambda_1, 12\mu(\Theta\sigma)_1, 8\Gamma\sigma_1, 8\Psi_1, 4\Psi\mu_1, 8\Lambda(\Theta\sigma)_1, 4\Theta(\mu\sigma)_1, 2\Delta\Lambda_1, \Sigma\sigma_1$
$\Theta\sigma\otimes\Theta\sigma$	$80k, 24\Theta, 64\mu, 24\Theta_1, 16\Lambda, 20\sigma_1, 16\Delta, 4\Theta\Theta_1, 16\Psi, 8\mu_1, 4\Theta\sigma_1, 4\mu\Theta_1, 4\Gamma, 2\mu\sigma_1, 2\Lambda\Theta_1, (\Theta\sigma)_1$
$\Theta\sigma\otimes\mu\sigma$	$120k, 40\Theta, 96\mu, 44\Theta_1, 24\Lambda, 40\sigma_1, 24\Delta, 8\Theta\Theta_1, 36\Psi, 24\mu_1, 8\Theta\sigma_1, 8\mu\Theta_1, 18\Gamma, 8\mu\sigma_1, 8\Lambda\Theta_1, 4\Theta\mu_1, 2(\Theta\sigma)_1, 2\Lambda_1, \Lambda\sigma_1, 2\Sigma, \Delta\sigma_1$
$\Theta\sigma\otimes\Lambda\sigma$	$176k, 64\Theta, 128\mu, 64\Theta_1, 32\Lambda, 66\sigma_1, 32\Delta, 12\Theta\Theta_1, 48\Psi, 40\mu_1, 16\Theta\sigma_1, 8\mu\Theta_1, 24\Gamma, 20\mu\sigma_1, 12\Lambda\Theta_1, 8\Theta\mu_1, 4(\Theta\sigma)_1, 8\Lambda_1, 4\Lambda\sigma_1, 4\Sigma, 2\sigma_2, 4\Delta\sigma_1, 2\Theta\Lambda_1, \Psi\sigma_1$
$\Theta\sigma\otimes\Delta\sigma$	$464k, 192\Theta, 304\mu, 200\Theta_1, 88\Lambda, 138\sigma_1, 64\Delta, 56\Theta\Theta_1, 112\Psi, 120\mu_1, 24\Theta\sigma_1, 64\mu\Theta_1, 8\Theta_2, 64\Gamma, 40\mu\sigma_1, 48\Lambda\Theta_1, 32\Theta\mu_1, 24(\Theta\sigma)_1, 20\Lambda_1, 8\Delta\Theta_1, 4\Lambda\sigma_1, 12\Sigma, 2\sigma_2, 8\Psi\Theta_1, 8\mu\mu_1, 12\Delta\sigma_1, 4\Delta_1, 8\Theta\Lambda_1, 2\Theta(\Theta\sigma)_1, 3\Psi\sigma_1, 4\Lambda\mu_1, 2(\mu\sigma)_1, 2\Gamma\Theta_1, \mu(\Theta\sigma)_1$

TABLE 2-3 (CONTINUED)

product	composition factors
$\Theta\sigma\otimes\Psi\sigma$	$428k, 206\Theta, 192\mu, 208\Theta_1, 72\Lambda, 128\sigma_1, 24\Delta, 72\Theta\Theta_1, 76\Psi, 160\mu_1, 18\Theta\sigma_1, 72\mu\Theta_1, 12\Theta_2, 64\Gamma, 64\mu\sigma_1, 56\Lambda\Theta_1, 48\Theta\mu_1, 40(\Theta\sigma)_1, 48\Lambda_1, 8\Delta\Theta_1, 8\Lambda\sigma_1, 18\Sigma, 12\sigma_2, 12\Psi\Theta_1, 16\mu\mu_1, 24\Delta\sigma_1, 8\Delta_1, 16\Theta\Lambda_1, 4\Theta(\Theta\sigma)_1, 12\Psi\sigma_1, 8\Lambda\mu_1, 8(\mu\sigma)_1, 4\Gamma\Theta_1, 4\mu\Lambda_1, 4\mu(\Theta\sigma)_1, 2\Gamma\sigma_1, 2\Psi_1, \Lambda(\Theta\sigma)_1$
$\Theta\sigma\otimes\Gamma\sigma$	$1216k, 496\Theta, 528\mu, 576\Theta_1, 176\Lambda, 376\sigma_1, 80\Delta, 176\Theta\Theta_1, 184\Psi, 488\mu_1, 72\Theta\sigma_1, 176\mu\Theta_1, 32\Theta_2, 136\Gamma, 196\mu\sigma_1, 120\Lambda\Theta_1, 128\Theta\mu_1, 108(\Theta\sigma)_1, 160\Lambda_1, 20\Delta\Theta_1, 32\Lambda\sigma_1, 32\Sigma, 52\sigma_2, 24\Psi\Theta_1, 72\mu\mu_1, 64\Delta\sigma_1, 24\Delta_1, 44\Theta\Lambda_1, 12\Theta(\Theta\sigma)_1, 44\Psi\sigma_1, 24\Lambda\mu_1, 36(\mu\sigma)_1, 4\Theta\sigma_2, 8\Gamma\Theta_1, 8\Delta\mu_1, 20\mu\Lambda_1, 12\mu(\Theta\sigma)_1, 12\Gamma\sigma_1, 8\Psi_1, 4\Psi\mu_1, 2\mu\sigma_2, 6\Lambda(\Theta\sigma)_1, 2\Lambda\Lambda_1, 4\Theta(\mu\sigma)_1, (\Lambda\sigma)_1, 2\Delta\Lambda_1, \Sigma\sigma_1$
$\Theta\sigma\otimes S$	$4480k, 1392\Theta, 1748\mu, 2096\Theta_1, 576\Lambda, 1168\sigma_1, 320\Delta, 576\Theta\Theta_1, 496\Psi, 2064\mu_1, 272\Theta\sigma_1, 752\mu\Theta_1, 288\Theta_2, 304\Gamma, 560\mu\sigma_1, 336\Lambda\Theta_1, 432\Theta\mu_1, 460(\Theta\sigma)_1, 652\Lambda_1, 136\Delta\Theta_1, 108\Lambda\sigma_1, 64\Sigma, 288\sigma_2, 32\Theta\Theta_2, 156\Psi\Theta_1, 368\mu\mu_1, 160\Delta\sigma_1, 224\Delta_1, 144\Theta\Lambda_1, 84\Theta(\Theta\sigma)_1, 144\Psi\sigma_1, 128\Lambda\mu_1, 20\mu\Theta_2, 184(\mu\sigma)_1, 32\Theta\sigma_2, 64\Gamma\Theta_1, 64\Delta\mu_1, 24\Lambda\Theta_2, 104\mu\Lambda_1, 96\mu(\Theta\sigma)_1, 24\Theta\Delta_1, 12\sigma_1\Theta_2, 56\Gamma\sigma_1, 96\Psi_1, 40\Psi\mu_1, 20\mu\sigma_2, 44\Lambda(\Theta\sigma)_1, 16\Lambda\Lambda_1, 16\Theta_1\sigma_2, 8\Sigma\Theta_1, 32\Theta(\mu\sigma)_1, 16\mu_2, 16(\Lambda\sigma)_1, 20\Delta\Lambda_1, 8\mu\Delta_1, 12\Delta(\Theta\sigma)_1, 8\Sigma\sigma_1, 8\Gamma\mu_1, 8\Theta\Psi_1, 8\mu(\mu\sigma)_1, 6\Psi\Lambda_1, 6\Psi(\Theta\sigma)_1, 4\Lambda\Delta_1, 4\Delta\sigma_2, 4\Gamma_1, 4(\Delta\sigma)_1, 3\Theta(\Lambda\sigma)_1, 2\Lambda(\mu\sigma)_1, 2\mu\Psi_1, \Gamma(\Theta\sigma)_1$
$\mu\sigma\otimes\mu\sigma$	$256k, 120\Theta, 168\mu, 104\Theta_1, 52\Lambda, 86\sigma_1, 32\Delta, 32\Theta\Theta_1, 64\Psi, 64\mu_1, 16\Theta\sigma_1, 24\mu\Theta_1, 40\Gamma, 28\mu\sigma_1, 28\Lambda\Theta_1, 20\Theta\mu_1, 10(\Theta\sigma)_1, 12\Lambda_1, 4\Lambda\sigma_1, 8\Sigma, 2\sigma_2, 4\mu\mu_1, 8\Delta\sigma_1, 4\Theta\Lambda_1, 2\Psi\sigma_1, 2\Lambda\mu_1, (\mu\sigma)_1$
$\mu\sigma\otimes\Lambda\sigma$	$400k, 184\Theta, 240\mu, 168\Theta_1, 72\Lambda, 136\sigma_1, 48\Delta, 52\Theta\Theta_1, 88\Psi, 112\mu_1, 26\Theta\sigma_1, 36\mu\Theta_1, 56\Gamma, 56\mu\sigma_1, 40\Lambda\Theta_1, 36\Theta\mu_1, 20(\Theta\sigma)_1, 28\Lambda_1, 8\Lambda\sigma_1, 12\Sigma, 8\sigma_2, 8\mu\mu_1, 18\Delta\sigma_1, 10\Theta\Lambda_1, 8\Psi\sigma_1, 4\Lambda\mu_1, 4(\mu\sigma)_1, \Theta\sigma_2, 2\mu\Lambda_1, \Gamma\sigma_1$
$\mu\sigma\otimes\Delta\sigma$	$1152k, 456\Theta, 624\mu, 520\Theta_1, 192\Lambda, 344\sigma_1, 120\Delta, 148\Theta\Theta_1, 220\Psi, 372\mu_1, 64\Theta\sigma_1, 172\mu\Theta_1, 32\Theta_2, 136\Gamma, 136\mu\sigma_1, 112\Lambda\Theta_1, 92\Theta\mu_1, 84(\Theta\sigma)_1, 96\Lambda_1, 24\Delta\Theta_1, 20\Lambda\sigma_1, 28\Sigma, 24\sigma_2, 28\Psi\Theta_1, 40\mu\mu_1, 42\Delta\sigma_1, 20\Delta_1, 26\Theta\Lambda_1, 10\Theta(\Theta\sigma)_1, 20\Psi\sigma_1, 16\Lambda\mu_1, 18(\mu\sigma)_1, \Theta\sigma_2, 8\Gamma\Theta_1, 4\Delta\mu_1, 6\mu\Lambda_1, 8\mu(\Theta\sigma)_1, 3\Gamma\sigma_1, 4\Psi_1, 2\Psi\mu_1, 2\Lambda(\Theta\sigma)_1, \Theta(\mu\sigma)_1$
$\mu\sigma\otimes\Psi\sigma$	$1284k, 440\Theta, 544\mu, 616\Theta_1, 168\Lambda, 364\sigma_1, 96\Delta, 168\Theta\Theta_1, 168\Psi, 512\mu_1, 76\Theta\sigma_1, 216\mu\Theta_1, 52\Theta_2, 108\Gamma, 182\mu\sigma_1, 108\Lambda\Theta_1, 112\Theta\mu_1, 126(\Theta\sigma)_1, 164\Lambda_1, 36\Delta\Theta_1, 32\Lambda\sigma_1, 24\Sigma, 56\sigma_2, 40\Psi\Theta_1, 72\mu\mu_1, 56\Delta\sigma_1, 35\Delta_1, 40\Theta\Lambda_1, 20\Theta(\Theta\sigma)_1, 42\Psi\sigma_1, 24\Lambda\mu_1, 36(\mu\sigma)_1, 4\Theta\sigma_2, 12\Gamma\Theta_1, 8\Delta\mu_1, 20\mu\Lambda_1, 18\mu(\Theta\sigma)_1, 12\Gamma\sigma_1, 10\Psi_1, 4\Psi\mu_1, 2\mu\sigma_2, 8\Lambda(\Theta\sigma)_1, 2\Lambda\Lambda_1, \Theta_1\sigma_2, 4\Theta(\mu\sigma)_1, (\Lambda\sigma)_1, 2\Delta\Lambda_1, \Sigma\sigma_1$
$\mu\sigma\otimes\Gamma\sigma$	$3360k, 1040\Theta, 1376\mu, 1572\Theta_1, 392\Lambda, 952\sigma_1, 264\Delta, 392\Theta\Theta_1, 408\Psi, 1440\mu_1, 232\Theta\sigma_1, 520\mu\Theta_1, 144\Theta_2, 248\Gamma, 468\mu\sigma_1, 244\Lambda\Theta_1, 280\Theta\mu_1, 320(\Theta\sigma)_1, 476\Lambda_1, 92\Delta\Theta_1, 96\Lambda\sigma_1, 52\Sigma, 200\sigma_2, 98\Psi\Theta_1, 240\mu\mu_1, 138\Delta\sigma_1, 112\Delta_1, 100\Theta\Lambda_1, 56\Theta(\Theta\sigma)_1, 124\Psi\sigma_1, 60\Lambda\mu_1, 120(\mu\sigma)_1, 18\Theta\sigma_2, 24\Gamma\Theta_1, 44\Delta\mu_1, 76\mu\Lambda_1, 48\mu(\Theta\sigma)_1, 46\Gamma\sigma_1, 48\Psi_1, 24\Psi\mu_1, 16\mu\sigma_2, 30\Lambda(\Theta\sigma)_1, 10\Lambda\Lambda_1, 8\Theta_1\sigma_2, 22\Theta(\mu\sigma)_1, 8\mu_2, 10(\Lambda\sigma)_1, 14\Delta\Lambda_1, \Lambda\sigma_2, 8\Sigma\sigma_1, 4\Gamma\mu_1, 4\mu(\mu\sigma)_1, 4\Psi\Lambda_1, 3\Delta\sigma_2, 2\Gamma_1, 2\Theta(\Lambda\sigma)_1, \Lambda(\mu\sigma)_1$

TABLE 2-3 (CONTINUED)

product	composition factors
$\mu\sigma \otimes S$	$11768k, 3248\Theta, 4364\mu, 5384\Theta_1, 1384\Lambda, 2928\sigma_1, 864\Delta, 1396\Theta\Theta_1, 1152\Psi, 5920\mu_1, 752\Theta\sigma_1, 1928\mu\Theta_1, 952\Theta_2, 624\Gamma, 1352\mu\sigma_1, 752\Lambda\Theta_1, 1200\Theta\mu_1, 1204(\Theta\sigma)_1, 1840\Lambda_1, 376\Delta\Theta_1, 296\Lambda\sigma_1, 120\Sigma, 1000\sigma_2, 144\Theta\Theta_2, 408\Psi\Theta_1, 1136\mu\mu_1, 344\Delta\sigma_1, 776\Delta_1, 380\Theta\Lambda_1, 240\Theta(\Theta\sigma)_1, 358\Psi\sigma_1, 384\Lambda\mu_1, 92\mu\Theta_2, 570(\mu\sigma)_1, 144\Theta\sigma_2, 172\Gamma\Theta_1, 208\Delta\mu_1, 88\Lambda\Theta_2, 356\mu\Lambda_1, 294\mu(\Theta\sigma)_1, 112\Theta\Delta_1, 56\sigma_1\Theta_2, 160\Gamma\sigma_1, 388\Psi_1, 144\Psi\mu_1, 116\mu\sigma_2, 132\Lambda(\Theta\sigma)_1, 72\Lambda\Lambda_1, 94\Theta_1\sigma_2, 24\Sigma\Theta_1, 104\Theta(\mu\sigma)_1, 88\mu_2, 88(\Lambda\sigma)_1, 76\Delta\Lambda_1, 44\mu\Delta_1, 48\Delta(\Theta\sigma)_1, 18\Lambda\sigma_2, 26\Sigma\sigma_1, 40\Gamma\mu_1, 8\sigma_1\sigma_2, 48\Theta\Psi_1, 44\mu(\mu\sigma)_1, 36\Psi\Lambda_1, 30\Psi(\Theta\sigma)_1, 24\Lambda\Delta_1, 24\Delta\sigma_2, 28\Gamma_1, 8\Theta\mu_2, 22(\Delta\sigma)_1, 8\Theta\Theta_1\sigma_2, 18\Theta(\Lambda\sigma)_1, 12\Lambda(\mu\sigma)_1, 14\mu\Psi_1, 6\Psi\sigma_2, 4\Sigma\mu_1, 6\mu_1\sigma_2, 6\Gamma(\Theta\sigma)_1, 6\Gamma\Lambda_1, 3\mu(\Lambda\sigma)_1, 3\mu\Theta_1\sigma_2, 4\Lambda\Psi_1, 4\Delta(\mu\sigma)_1, 2\Theta\Gamma_1, 2(\Psi\sigma)_1, \Psi(\mu\sigma)_1$
$\Lambda\sigma \otimes \Lambda\sigma$	$672k, 288\Theta, 368\mu, 288\Theta_1, 112\Lambda, 224\sigma_1, 64\Delta, 88\Theta\Theta_1, 128\Psi, 208\mu_1, 40\Theta\sigma_1, 64\mu\Theta_1, 80\Gamma, 104\mu\sigma_1, 80\Lambda\Theta_1, 64\Theta\mu_1, 38(\Theta\sigma)_1, 64\Lambda_1, 16\Lambda\sigma_1, 16\Sigma, 20\sigma_2, 24\mu\mu_1, 32\Delta\sigma_1, 20\Theta\Lambda_1, 20\Psi\sigma_1, 8\Lambda\mu_1, 12(\mu\sigma)_1, 4\Theta\sigma_2, 8\mu\Lambda_1, 4\Gamma\sigma_1, 2\mu\sigma_2, 2\Lambda\Lambda_1, (\Lambda\sigma)_1$
$\Lambda\sigma \otimes \Delta\sigma$	$1812k, 672\Theta, 912\mu, 824\Theta_1, 272\Lambda, 548\sigma_1, 176\Delta, 224\Theta\Theta_1, 312\Psi, 632\mu_1, 112\Theta\sigma_1, 256\mu\Theta_1, 52\Theta_2, 192\Gamma, 240\mu\sigma_1, 160\Lambda\Theta_1, 144\Theta\mu_1, 144(\Theta\sigma)_1, 180\Lambda_1, 36\Delta\Theta_1, 40\Lambda\sigma_1, 40\Sigma, 60\sigma_2, 40\Psi\Theta_1, 72\mu\mu_1, 72\Delta\sigma_1, 36\Delta_1, 44\Theta\Lambda_1, 20\Theta(\Theta\sigma)_1, 46\Psi\sigma_1, 24\Lambda\mu_1, 36(\mu\sigma)_1, 4\Theta\sigma_2, 12\Gamma\Theta_1, 6\Delta\mu_1, 18\mu\Lambda_1, 18\mu(\Theta\sigma)_1, 12\Gamma\sigma_1, 10\Psi_1, 4\Psi\mu_1, 2\mu\sigma_2, 8\Lambda(\Theta\sigma)_1, \Theta_1\sigma_2, 4\Theta(\mu\sigma)_1, 2\Delta\Lambda_1, \Sigma\sigma_1$
$\Lambda\sigma \otimes \Psi\sigma$	$2008k, 624\Theta, 832\mu, 960\Theta_1, 240\Lambda, 560\sigma_1, 160\Delta, 240\Theta\Theta_1, 248\Psi, 832\mu_1, 136\Theta\sigma_1, 336\mu\Theta_1, 88\Theta_2, 152\Gamma, 280\mu\sigma_1, 152\Lambda\Theta_1, 160\Theta\mu_1, 200(\Theta\sigma)_1, 272\Lambda_1, 84\Delta\Theta_1, 54\Lambda\sigma_1, 32\Sigma, 104\sigma_2, 60\Psi\Theta_1, 128\mu\mu_1, 80\Delta\sigma_1, 84\Delta_1, 56\Theta\Lambda_1, 36\Theta(\Theta\sigma)_1, 72\Psi\sigma_1, 32\Lambda\mu_1, 64(\mu\sigma)_1, 8\Theta\sigma_2, 16\Gamma\Theta_1, 24\Delta\mu_1, 40\mu\Lambda_1, 32\mu(\Theta\sigma)_1, 28\Gamma\sigma_1, 20\Psi_1, 8\Psi\mu_1, 8\mu\sigma_2, 20\Lambda(\Theta\sigma)_1, 4\Lambda\Lambda_1, 4\Theta_1\sigma_2, 12\Theta(\mu\sigma)_1, 4(\Lambda\sigma)_1, 8\Delta\Lambda_1, 4\Sigma\sigma_1, 2\Psi\Lambda_1, 2\Delta\sigma_2, \Theta(\Lambda\sigma)_1$
$\Lambda\sigma \otimes \Gamma\sigma$	$5368k, 1504\Theta, 2144\mu, 2488\Theta_1, 576\Lambda, 1478\sigma_1, 448\Delta, 578\Theta\Theta_1, 816\Psi, 2424\mu_1, 400\Theta\sigma_1, 832\mu\Theta_1, 264\Theta_2, 352\Gamma, 752\mu\sigma_1, 352\Lambda\Theta_1, 416\Theta\mu_1, 512(\Theta\sigma)_1, 804\Lambda_1, 168\Delta\Theta_1, 160\Lambda\sigma_1, 72\Sigma, 382\sigma_2, 140\Psi\Theta_1, 432\mu\mu_1, 200\Delta\sigma_1, 218\Delta_1, 152\Theta\Lambda_1, 104\Theta(\Theta\sigma)_1, 202\Psi\sigma_1, 96\Lambda\mu_1, 216(\mu\sigma)_1, 32\Theta\sigma_2, 36\Gamma\Theta_1, 96\Delta\mu_1, 152\mu\Lambda_1, 82\mu(\Theta\sigma)_1, 88\Gamma\sigma_1, 108\Psi_1, 32\Psi\mu_1, 44\mu\sigma_2, 60\Lambda(\Theta\sigma)_1, 20\Lambda\Lambda_1, 26\Theta_1\sigma_2, 48\Theta(\mu\sigma)_1, 24\mu_2, 28(\Lambda\sigma)_1, 36\Delta\Lambda_1, 4\Lambda\sigma_2, 14\Sigma\sigma_1, 8\Gamma\mu_1, 2\sigma_1\sigma_2, 12\mu(\mu\sigma)_1, 14\Psi\Lambda_1, 12\Delta\sigma_2, 8\Gamma_1, 8\Theta(\Lambda\sigma)_1, 4\Lambda(\mu\sigma)_1, 3\Psi\sigma_2, 2\mu_1\sigma_2, 2\Gamma\Lambda_1, \mu(\Lambda\sigma)_1$

TABLE 2-3 (CONTINUED)

product	composition factors
$\Lambda\sigma \otimes S$	$19168k, 5064\Theta, 6888\mu, 8640\Theta_1, 2124\Lambda, 4640\sigma_1, 1408\Delta, 2160\Theta\Theta_1, 1760\Psi, 10144\mu_1, 1248\Theta\sigma_1, 3072\mu\Theta_1, 1696\Theta_2, 928\Gamma, 2112\mu\sigma_1, 1144\Lambda\Theta_1, 2016\Theta\mu_1, 1944(\Theta\sigma)_1, 3144\Lambda_1, 624\Delta\Theta_1, 488\Lambda\sigma_1, 176\Sigma, 1688\sigma_2, 264\Theta\Theta_2, 648\Psi\Theta_1, 2016\mu\mu_1, 520\Delta\sigma_1, 1440\Delta_1, 640\Theta\Lambda_1, 408\Theta(\Theta\sigma)_1, 568\Psi\sigma_1, 656\Lambda\mu_1, 168\mu\Theta_2, 1016(\mu\sigma)_1, 284\Theta\sigma_2, 264\Gamma\Theta_1, 384\Delta\mu_1, 140\Lambda\Theta_2, 672\mu\Lambda_1, 504\mu(\Theta\sigma)_1, 216\Theta\Delta_1, 104\sigma_1\Theta_2, 272\Gamma\sigma_1, 784\Psi_1, 272\Psi\mu_1, 264\mu\sigma_2, 232\Lambda(\Theta\sigma)_1, 140\Lambda\Lambda_1, 216\Theta_1\sigma_2, 36\Sigma\Theta_1, 192\Theta(\mu\sigma)_1, 192\mu_2, 192(\Lambda\sigma)_1, 152\Delta\Lambda_1, 96\mu\Delta_1, 82\Delta(\Theta\sigma)_1, 52\Lambda\sigma_2, 48\Sigma\sigma_1, 80\Gamma\mu_1, 32\sigma_1\sigma_2, 108\Theta\Psi_1, 96\mu(\mu\sigma)_1, 84\Psi\Lambda_1, 60\Psi(\Theta\sigma)_1, 52\Lambda\Delta_1, 56\Delta\sigma_2, 72\Gamma_1, 24\Theta\mu_2, 48(\Delta\sigma)_1, 26\Theta\Theta_1\sigma_2, 42\Theta(\Lambda\sigma)_1, 28\Lambda(\mu\sigma)_1, 36\mu\Psi_1, 24\Psi\sigma_2, 8\Sigma\mu_1, 24\mu_1\sigma_2, 14\Gamma(\Theta\sigma)_1, 20\Gamma\Lambda_1, 12\mu(\Lambda\sigma)_1, 12\mu\Theta_1\sigma_2, 14\Lambda\Psi_1, 12\Delta(\mu\sigma)_1, 4\Gamma\sigma_2, 8\Theta\Gamma_1, 8(\Psi\sigma)_1, 3\Lambda\Theta_1\sigma_2, 4\Psi(\mu\sigma)_1, 2\Theta\mu_1\sigma_2, 2\Sigma\Lambda_1, \Delta(\Lambda\sigma)_1$
$\Delta\sigma \otimes \Delta\sigma$	$5072k, 1816\Theta, 2418\mu, 2352\Theta_1, 800\Lambda, 1400\sigma_1, 448\Delta, 688\Theta\Theta_1, 784\Psi, 1920\mu_1, 272\Theta\sigma_1, 856\mu\Theta_1, 256\Theta_2, 472\Gamma, 588\mu\sigma_1, 456\Lambda\Theta_1, 456\Theta\mu_1, 448(\Theta\sigma)_1, 536\Lambda_1, 136\Delta\Theta_1, 96\Lambda\sigma_1, 96\Sigma, 160\sigma_2, 32\Theta\Theta_2, 178\Psi\Theta_1, 280\mu\mu_1, 178\Delta\sigma_1, 168\Delta_1, 136\Theta\Lambda_1, 64\Theta(\Theta\sigma)_1, 112\Psi\sigma_1, 128\Lambda\mu_1, 24\mu\Theta_2, 124(\mu\sigma)_1, 20\Theta\sigma_2, 72\Gamma\Theta_1, 32\Delta\mu_1, 28\Lambda\Theta_2, 52\mu\Lambda_1, 68\mu(\Theta\sigma)_1, 20\Theta\Delta_1, 10\sigma_1\Theta_2, 28\Gamma\sigma_1, 48\Psi_1, 24\Psi\mu_1, 6\mu\sigma_2, 20\Lambda(\Theta\sigma)_1, 8\Lambda\Lambda_1, 4\Theta_1\sigma_2, 8\Sigma\Theta_1, 12\Theta(\mu\sigma)_1, 8\mu_2, 4(\Lambda\sigma)_1, 4\Delta\Lambda_1, 4\mu\Delta_1, 8\Delta(\Theta\sigma)_1, 2\Sigma\sigma_1, 4\Gamma\mu_1, 4\Theta\Psi_1, 2\mu(\mu\sigma)_1, 2\Psi(\Theta\sigma)_1, 2\Lambda\Delta_1, (\Delta\sigma)_1$
$\Delta\sigma \otimes \Psi\sigma$	$5680k, 1764\Theta, 2260\mu, 2672\Theta_1, 752\Lambda, 1504\sigma_1, 416\Delta, 744\Theta\Theta_1, 640\Psi, 2544\mu_1, 344\Theta\sigma_1, 960\mu\Theta_1, 368\Theta_2, 376\Gamma, 712\mu\sigma_1, 424\Lambda\Theta_1, 552\Theta\mu_1, 580(\Theta\sigma)_1, 796\Lambda_1, 168\Delta\Theta_1, 140\Lambda\sigma_1, 76\Sigma, 352\sigma_2, 52\Theta\Theta_2, 204\Psi\Theta_1, 432\mu\mu_1, 198\Delta\sigma_1, 272\Delta_1, 180\Theta\Lambda_1, 100\Theta(\Theta\sigma)_1, 178\Psi\sigma_1, 168\Lambda\mu_1, 36\mu\Theta_2, 216(\mu\sigma)_1, 48\Theta\sigma_2, 88\Gamma\Theta_1, 64\Delta\mu_1, 40\Lambda\Theta_2, 120\mu\Lambda_1, 120\mu(\Theta\sigma)_1, 36\Theta\Delta_1, 20\sigma_1\Theta_2, 84\Gamma\sigma_1, 104\Psi_1, 40\Psi\mu_1, 28\mu\sigma_2, 48\Lambda(\Theta\sigma)_1, 20\Lambda\Lambda_1, 20\Theta_1\sigma_2, 12\Sigma\Theta_1, 32\Theta(\mu\sigma)_1, 16\mu_2, 20(\Lambda\sigma)_1, 20\Delta\Lambda_1, 8\mu\Delta_1, 18\Delta(\Theta\sigma)_1, 2\Lambda\sigma_2, 8\Sigma\sigma_1, 8\Gamma\mu_1, 10\Theta\Psi_1, 8\mu(\mu\sigma)_1, 8\Psi\Lambda_1, 8\Psi(\Theta\sigma)_1, 4\Lambda\Delta_1, 4\Delta\sigma_2, 4\Gamma_1, 4(\Delta\sigma)_1, \Theta\Theta_1\sigma_2, 3\Theta(\Lambda\sigma)_1, 2\Lambda(\mu\sigma)_1, 2\mu\Psi_1, \Gamma(\Theta\sigma)_1$
$\Delta\sigma \otimes \Gamma\sigma$	$15072k, 4336\Theta, 5740\mu, 6928\Theta_1, 1784\Lambda, 3924\sigma_1, 1120\Delta, 1796\Theta\Theta_1, 1560\Psi, 7272\mu_1, 992\Theta\sigma_1, 2392\mu\Theta_1, 1056\Theta_2, 880\Gamma, 1880\mu\sigma_1, 992\Lambda\Theta_1, 1488\Theta\mu_1, 1508(\Theta\sigma)_1, 2308\Lambda_1, 448\Delta\Theta_1, 400\Lambda\sigma_1, 176\Sigma, 1198\sigma_2, 144\Theta\Theta_2, 476\Psi\Theta_1, 1344\mu\mu_1, 498\Delta\sigma_1, 856\Delta_1, 488\Theta\Lambda_1, 288\Theta(\Theta\sigma)_1, 482\Psi\sigma_1, 440\Lambda\mu_1, 92\mu\Theta_2, 578(\mu\sigma)_1, 168\Theta\sigma_2, 192\Gamma\Theta_1, 240\Delta\mu_1, 68\Lambda\Theta_2, 428\mu\Lambda_1, 336\mu(\Theta\sigma)_1, 112\Theta\Delta_1, 56\sigma_1\Theta_2, 208\Gamma\sigma_1, 416\Psi_1, 156\Psi\mu_1, 140\mu\sigma_2, 160\Lambda(\Theta\sigma)_1, 80\Lambda\Lambda_1, 100\Theta_1\sigma_2, 24\Sigma\Theta_1, 120\Theta(\mu\sigma)_1, 88\mu_2, 96(\Lambda\sigma)_1, 88\Delta\Lambda_1, 44\mu\Delta_1, 48\Delta(\Theta\sigma)_1, 20\Lambda\sigma_2, 32\Sigma\sigma_1, 40\Gamma\mu_1, 10\sigma_1\sigma_2, 48\Theta\Psi_1, 44\mu(\mu\sigma)_1, 38\Psi\Lambda_1, 30\Psi(\Theta\sigma)_1, 28\Lambda\Delta_1, 28\Delta\sigma_2, 28\Gamma_1, 8\Theta\mu_2, 22(\Delta\sigma)_1, 8\Theta\Theta_1\sigma_2, 20\Theta(\Lambda\sigma)_1, 12\Lambda(\mu\sigma)_1, 14\mu\Psi_1, 7\Psi\sigma_2, 4\Sigma\mu_1, 8\mu_1\sigma_2, 6\Gamma(\Theta\sigma)_1, 6\Gamma\Lambda_1, 3\mu(\Lambda\sigma)_1, 3\mu\Theta_1\sigma_2, 4\Lambda\Psi_1, 4\Delta(\mu\sigma)_1, 2\Theta\Gamma_1, 2(\Psi\sigma)_1, \Psi(\mu\sigma)_1$

TABLE 2-3 (CONTINUED)

product	composition factors
$\Delta\sigma \otimes S$	$54832k, 14336\Theta, 19008\mu, 24192\Theta_1, 6200\Lambda, 12592\sigma_1, 3788\Delta, 6272\Theta\Theta_1, 4712\Psi, 30592\mu_1, 3228\Theta\sigma_1, 8664\mu\Theta_1, 6080\Theta_2, 2432\Gamma, 5408\mu\sigma_1, 3184\Lambda\Theta_1, 6272\Theta\mu_1, 5520(\Theta\sigma)_1, 9088\Lambda_1, 1720\Delta\Theta_1, 1296\Lambda\sigma_1, 448\Sigma, 5872\sigma_2, 1040\Theta\Theta_2, 1920\Psi\Theta_1, 6272\mu\mu_1, 1312\Delta\sigma_1, 4912\Delta_1, 1872\Theta\Lambda_1, 1168\Theta(\Theta\sigma)_1, 1400\Psi\sigma_1, 2144\Lambda\mu_1, 896\mu\Theta_2, 3176(\mu\sigma)_1, 936\Theta\sigma_2, 864\Gamma\Theta_1, 144\Theta_1\Theta_2, 1168\Delta\mu_1, 568\Lambda\Theta_2, 1904\mu\Lambda_1, 1600\mu(\Theta\sigma)_1, 824\Theta\Delta_1, 544\sigma_1\Theta_2, 652\Gamma\sigma_1, 2840\Psi_1, 896\Psi\mu_1, 800\mu\sigma_2, 628\Lambda(\Theta\sigma)_1, 92\Delta\Theta_2, 440\Lambda\Lambda_1, 720\Theta_1\sigma_2, 136\Sigma\Theta_1, 568\Theta(\mu\sigma)_1, 800\mu_2, 600(\Lambda\sigma)_1, 412\Delta\Lambda_1, 480\mu\Delta_1, 308\Delta(\Theta\sigma)_1, 68\Psi\Theta_2, 152\Lambda\sigma_2, 112\Sigma\sigma_1, 268\Gamma\mu_1, 56\Theta\sigma_1\Theta_2, 112\sigma_1\sigma_2, 112\mu_1\Theta_2, 416\Theta\Psi_1, 368\mu(\mu\sigma)_1, 240\Psi\Lambda_1, 240\Psi(\Theta\sigma)_1, 216\Lambda\Delta_1, 184\Delta\sigma_2, 312\Gamma_1, 96\Theta\mu_2, 240(\Delta\sigma)_1, 24\Gamma\Theta_2, 60\Theta\Theta_1\sigma_2, 124\Theta(\Lambda\sigma)_1, 44\Delta\Delta_1, 128\Lambda(\mu\sigma)_1, 48\mu\sigma_1\Theta_2, 176\mu\Psi_1, 64\Psi\sigma_2, 48\Lambda_1\Theta_2, 32\Sigma\mu_1, 88\mu_1\sigma_2, 72\Gamma(\Theta\sigma)_1, 58\Gamma\Lambda_1, 8\Theta\sigma_1\sigma_2, 40\mu(\Lambda\sigma)_1, 16\mu\mu_2, 30\Lambda\sigma_1\Theta_2, 40\mu\Theta_1\sigma_2, 52\Lambda\Psi_1, 24\Psi\Delta_1, 48\Delta(\mu\sigma)_1, 8\Theta_1\mu_2, 8\Gamma\sigma_2, 8(\Theta\sigma)_2, 22\Theta(\Delta\sigma)_1, 8\Lambda_2, 32\Theta\Gamma_1, 40(\Psi\sigma)_1, 6\Lambda\Theta_1\sigma_2, 14\Delta\Psi_1, 4\Lambda(\Lambda\sigma)_1, 20\Psi(\mu\sigma)_1, 8\Lambda\mu_2, 8\sigma_1\mu_2, 6\Sigma(\Theta\sigma)_1, 4\Theta\mu_1\sigma_2, 4\Sigma\Lambda_1, 4\Gamma\Delta_1, 3\Delta\Theta_1\sigma_2, 4\Psi\Psi_1, 4\mu(\Delta\sigma)_1, 2\Delta(\Lambda\sigma)_1, 4\mu\Gamma_1, 2\Sigma_1, 2\Theta(\Psi\sigma)_1, 2\Gamma(\mu\sigma)_1, \Lambda(\Delta\sigma)_1$
$\Psi\sigma \otimes \Psi\sigma$	$6720k, 1688\Theta, 2464\mu, 3072\Theta_1, 764\Lambda, 1654\sigma_1, 512\Delta, 768\Theta\Theta_1, 608\Psi, 3424\mu_1, 448\Theta\sigma_1, 1104\mu\Theta_1, 576\Theta_2, 288\Gamma, 752\mu\sigma_1, 384\Lambda\Theta_1, 640\Theta\mu_1, 696(\Theta\sigma)_1, 1072\Lambda_1, 224\Delta\Theta_1, 176\Lambda\sigma_1, 48\Sigma, 608\sigma_2, 88\Theta\Theta_2, 240\Psi\Theta_1, 672\mu\mu_1, 176\Delta\sigma_1, 464\Delta_1, 208\Theta\Lambda_1, 144\Theta(\Theta\sigma)_1, 204\Psi\sigma_1, 224\Lambda\mu_1, 64\mu\Theta_2, 336(\mu\sigma)_1, 88\Theta\sigma_2, 104\Gamma\Theta_1, 128\Delta\mu_1, 60\Lambda\Theta_2, 216\mu\Lambda_1, 180\mu(\Theta\sigma)_1, 64\Theta\Delta_1, 38\sigma_1\Theta_2, 96\Gamma\sigma_1, 224\Psi_1, 80\Psi\mu_1, 80\mu\sigma_2, 80\Lambda(\Theta\sigma)_1, 40\Lambda\Lambda_1, 60\Theta_1\sigma_2, 18\Sigma\Theta_1, 84\Theta(\mu\sigma)_1, 48\mu_2, 58(\Lambda\sigma)_1, 48\Delta\Lambda_1, 24\mu\Delta_1, 32\Delta(\Theta\sigma)_1, 12\Lambda\sigma_2, 16\Sigma\sigma_1, 16\Gamma\mu_1, 6\sigma_1\sigma_2, 20\Theta\Psi_1, 24\mu(\mu\sigma)_1, 24\Psi\Lambda_1, 20\Psi(\Theta\sigma)_1, 8\Lambda\Delta_1, 18\Delta\sigma_2, 16\Gamma_1, 12(\Delta\sigma)_1, 4\Theta\Theta_1\sigma_2, 12\Theta(\Lambda\sigma)_1, 8\Lambda(\mu\sigma)_1, 8\mu\Psi_1, 4\Psi\sigma_2, 4\mu_1\sigma_2, 4\Gamma(\Theta\sigma)_1, 4\Gamma\Lambda_1, 2\mu(\Lambda\sigma)_1, 2\mu\Theta_1\sigma_2, 2\Lambda\Psi_1, (\Psi\sigma)_1$
$\Psi\sigma \otimes \Gamma\sigma$	$17848k, 4424\Theta, 6280\mu, 7984\Theta_1, 1900\Lambda, 4248\sigma_1, 1328\Delta, 1936\Theta\Theta_1, 1528\Psi, 9696\mu_1, 1192\Theta\sigma_1, 2824\mu\Theta_1, 1660\Theta_2, 736\Gamma, 1920\mu\sigma_1, 968\Lambda\Theta_1, 1872\Theta\mu_1, 1808(\Theta\sigma)_1, 3032\Lambda_1, 592\Delta\Theta_1, 464\Lambda\sigma_1, 128\Sigma, 1920\sigma_2, 264\Theta\Theta_2, 600\Psi\Theta_1, 1968\mu\mu_1, 448\Delta\sigma_1, 1424\Delta_1, 600\Theta\Lambda_1, 392\Theta(\Theta\sigma)_1, 532\Psi\sigma_1, 632\Lambda\mu_1, 168\mu\Theta_2, 992(\mu\sigma)_1, 300\Theta\sigma_2, 248\Gamma\Theta_1, 384\Delta\mu_1, 140\Lambda\Theta_2, 692\mu\Lambda_1, 488\mu(\Theta\sigma)_1, 216\Theta\Delta_1, 104\sigma_1\Theta_2, 266\Gamma\sigma_1, 796\Psi_1, 272\Psi\mu_1, 296\mu\sigma_2, 228\Lambda(\Theta\sigma)_1, 148\Lambda\Lambda_1, 232\Theta_1\sigma_2, 36\Sigma\Theta_1, 192\Theta(\mu\sigma)_1, 200\mu_2, 208(\Lambda\sigma)_1, 160\Delta\Lambda_1, 96\mu\Delta_1, 82\Delta(\Theta\sigma)_1, 60\Lambda\sigma_2, 48\Sigma\sigma_1, 80\Gamma\mu_1, 40\sigma_1\sigma_2, 108\Theta\Psi_1, 96\mu(\mu\sigma)_1, 96\Psi\Lambda_1, 60\Psi(\Theta\sigma)_1, 52\Lambda\Delta_1, 64\Delta\sigma_2, 80\Gamma_1, 24\Theta\mu_2, 48(\Delta\sigma)_1, 26\Theta\Theta_1\sigma_2, 46\Theta(\Lambda\sigma)_1, 28\Lambda(\mu\sigma)_1, 36\mu\Psi_1, 28\Psi\sigma_2, 8\Sigma\mu_1, 28\mu_1\sigma_2, 14\Gamma(\Theta\sigma)_1, 24\Gamma\Lambda_1, 2\Theta\sigma_1\sigma_2, 16\mu(\Lambda\sigma)_1, 12\mu\Theta_1\sigma_2, 14\Lambda\Psi_1, 12\Delta(\mu\sigma)_1, 4\Gamma\sigma_2, 2\Lambda_2, 8\Theta\Gamma_1, 8(\Psi\sigma)_1, 3\Lambda\Theta_1\sigma_2, \Lambda(\Lambda\sigma)_1, 4\Psi(\mu\sigma)_1, 2\Theta\mu_1\sigma_2, 2\Sigma\Lambda_1, \Delta(\Lambda\sigma)_1$

TABLE 2-3 (CONTINUED)

product	composition factors
$\Psi\sigma \otimes S$	$66040k, 15876\Theta, 21264\mu, 27968\Theta_1, 6816\Lambda, 14048\sigma_1, 4360\Delta,$ $6960\Theta\Theta_1, 4892\Psi, 40528\mu_1, 3712\Theta\sigma_1, 9840\mu\Theta_1, 8808\Theta_2, 2324\Gamma,$ $5706\mu\sigma_1, 3320\Lambda\Theta_1, 6224\Theta\mu_1, 6408(\Theta\sigma)_1, 11616\Lambda_1, 2040\Delta\Theta_1,$ $1520\Lambda\sigma_1, 400\Sigma, 8944\sigma_2, 1556\Theta\Theta_2, 2200\Psi\Theta_1, 6736\mu\mu_1, 1280\Delta\sigma_1,$ $7352\Delta_1, 2432\Theta\Lambda_1, 1464\Theta(\Theta\sigma)_1, 1480\Psi\sigma_1, 2912\Lambda\mu_1, 1360\mu\Theta_2,$ $4472(\mu\sigma)_1, 1536\Theta\sigma_2, 992\Gamma\Theta_1, 264\Theta_1\Theta_2, 1680\Delta\mu_1, 736\Lambda\Theta_2,$ $2792\mu\Lambda_1, 2064\mu(\Theta\sigma)_1, 1296\Theta\Delta_1, 832\sigma_1\Theta_2, 736\Gamma\sigma_1, 4780\Psi_1,$ $1352\Psi\mu_1, 1448\mu\sigma_2, 788\Lambda(\Theta\sigma)_1, 168\Delta\Theta_2, 704\Lambda\Lambda_1, 1334\Theta_1\sigma_2,$ $160\Sigma\Theta_1, 856\Theta(\mu\sigma)_1, 1520\mu_2, 1032(\Lambda\sigma)_1, 620\Delta\Lambda_1, 864\mu\Delta_1,$ $416\Delta(\Theta\sigma)_1, 140\Psi\Theta_2, 352\Lambda\sigma_2, 134\Sigma\sigma_1, 464\Gamma\mu_1, 104\Theta\sigma_1\Theta_2, 304\sigma_1\sigma_2,$ $216\mu_1\Theta_2, 764\Theta\Psi_1, 648\mu(\mu\sigma)_1, 436\Psi\Lambda_1, 366\Psi(\Theta\sigma)_1, 368\Lambda\Delta_1,$ $296\Delta\sigma_2, 672\Gamma_1, 218\Theta\mu_2, 432(\Delta\sigma)_1, 36\Gamma\Theta_2, 164\Theta\Theta_1\sigma_2, 216\Theta(\Lambda\sigma)_1,$ $96\Delta\Delta_1, 232\Lambda(\mu\sigma)_1, 82\mu\sigma_1\Theta_2, 376\mu\Psi_1, 164\Psi\sigma_2, 108\Lambda_1\Theta_2,$ $56\Sigma\mu_1, 260\mu_1\sigma_2, 128\Gamma(\Theta\sigma)_1, 138\Gamma\Lambda_1, 32\Theta\sigma_1\sigma_2, 108\mu(\Lambda\sigma)_1, 68\mu\mu_2,$ $60\Lambda\sigma_1\Theta_2, 112\mu\Theta_1\sigma_2, 128\Lambda\Psi_1, 52\Psi\Delta_1, 96\Delta(\mu\sigma)_1, 24\Theta_1\mu_2, 36\Gamma\sigma_2,$ $26(\Theta\sigma)_2, 48\Theta(\Delta\sigma)_1, 32\Lambda_2, 68\Theta\Gamma_1, 112(\Psi\sigma)_1, 28\Lambda\Theta_1\sigma_2, 36\Delta\Psi_1,$ $16\Lambda(\Lambda\sigma)_1, 10\mu\sigma_1\sigma_2, 52\Psi(\mu\sigma)_1, 16\Lambda\mu_2, 8\Lambda_1\sigma_2, 8(\Theta\sigma)_1\sigma_2, 24\sigma_1\mu_2,$ $14\Sigma(\Theta\sigma)_1, 28\Theta\mu_1\sigma_2, 16\Sigma\Lambda_1, 8\Gamma\Delta_1, 12\Delta\Theta_1\sigma_2, 14\Psi\Psi_1, 12\mu(\Delta\sigma)_1,$ $16\Delta(\Lambda\sigma)_1, 16\mu\Gamma_1, 8\Sigma_1, 8\Theta(\Psi\sigma)_1, 8\Gamma(\mu\sigma)_1, 6\Theta\Lambda_2, 2\Sigma\sigma_2, 4\Lambda\Gamma_1,$ $4\Lambda(\Delta\sigma)_1, 4\mu\mu_1\sigma_2, 3\Psi\Theta_1\sigma_2, 3\Psi(\Lambda\sigma)_1, 2(\Gamma\sigma)_1, 2\Delta_1\sigma_2, 2\Gamma\Psi_1, \mu(\Psi\sigma)_1$
$\Gamma\sigma \otimes \Gamma\sigma$	$48320k, 11936\Theta, 16416\mu, 21024\Theta_1, 4944\Lambda, 11112\sigma_1, 3456\Delta, 5048\Theta\Theta_1,$ $3936\Psi, 27776\mu_1, 3136\Theta\sigma_1, 7328\mu\Theta_1, 4896\Theta_2, 1904\Gamma, 4984\mu\sigma_1,$ $2500\Lambda\Theta_1, 5600\Theta\mu_1, 4714(\Theta\sigma)_1, 8608\Lambda_1, 1552\Delta\Theta_1, 1216\Lambda\sigma_1, 336\Sigma,$ $6008\sigma_2, 840\Theta\Theta_2, 1536\Psi\Theta_1, 5824\mu\mu_1, 1152\Delta\sigma_1, 4384\Delta_1, 1776\Theta\Lambda_1,$ $1040\Theta(\Theta\sigma)_1, 1392\Psi\sigma_1, 1888\Lambda\mu_1, 480\mu\Theta_2, 2976(\mu\sigma)_1, 1040\Theta\sigma_2,$ $624\Gamma\Theta_1, 1120\Delta\mu_1, 368\Lambda\Theta_2, 2128\mu\Lambda_1, 1288\mu(\Theta\sigma)_1, 736\Theta\Delta_1, 296\sigma_1\Theta_2,$ $712\Gamma\sigma_1, 2768\Psi_1, 880\Psi\mu_1, 1012\mu\sigma_2, 616\Lambda(\Theta\sigma)_1, 512\Lambda\Lambda_1, 624\Theta_1\sigma_2,$ $68\Sigma\Theta_1, 568\Theta(\mu\sigma)_1, 816\mu_2, 708(\Lambda\sigma)_1, 488\Delta\Lambda_1, 352\mu\Delta_1, 216\Delta(\Theta\sigma)_1,$ $256\Lambda\sigma_2, 132\Sigma\sigma_1, 304\Gamma\mu_1, 196\sigma_1\sigma_2, 464\Theta\Psi_1, 376\mu(\mu\sigma)_1, 336\Psi\Lambda_1,$ $164\Psi(\Theta\sigma)_1, 208\Lambda\Delta_1, 208\Delta\sigma_2, 352\Gamma_1, 144\Theta\mu_2, 176(\Delta\sigma)_1, 120\Theta\Theta_1\sigma_2,$ $152\Theta(\Lambda\sigma)_1, 96\Lambda(\mu\sigma)_1, 168\mu\Psi_1, 120\Psi\sigma_2, 40\Sigma\mu_1, 148\mu_1\sigma_2, 40\Gamma(\Theta\sigma)_1,$ $104\Gamma\Lambda_1, 12\Theta\sigma_1\sigma_2, 76\mu(\Lambda\sigma)_1, 24\mu\mu_2, 66\mu\Theta_1\sigma_2, 80\Lambda\Psi_1, 64\Delta(\mu\sigma)_1,$ $28\Gamma\sigma_2, 16\Lambda_2, 60\Theta\Gamma_1, 56(\Psi\sigma)_1, 20\Lambda\Theta_1\sigma_2, 6\Lambda(\Lambda\sigma)_1, 6\mu\sigma_1\sigma_2,$ $32\Psi(\mu\sigma)_1, 6\Lambda\mu_2, 4\Lambda_1\sigma_2, 4(\Theta\sigma)_1\sigma_2, 12\sigma_1\mu_2, 20\Theta\mu_1\sigma_2, 12\Sigma\Lambda_1,$ $12\Delta(\Lambda\sigma)_1, 8\mu\Gamma_1, 4\Gamma(\mu\sigma)_1, 4\Theta\Lambda_2, 2\Sigma\sigma_2, 2\Lambda\Gamma_1, 2\mu\mu_1\sigma_2, 2\Psi(\Lambda\sigma)_1, (\Gamma\sigma)_1$

Table 2-3 (continued)

product	composition factors
$\Gamma\sigma \otimes S$	$181536\Delta, 43088\Theta, 56472\mu, 74904\Theta_1, 17888\Lambda, 37224\sigma_1, 11616\Delta, 18224\Theta_1, 12720\Psi, 116000\mu_1, 9896\Theta\sigma_1, 25728\mu\Theta_1, 25824\Theta_2, 5976\Gamma, 14936\mu\sigma_1, 8560\Lambda\Theta_1, 23712\Theta(\mu\sigma)_1, 16800(\Theta\sigma)_1, 33608\Lambda_1, 5376\Delta\Theta_1, 4044\Lambda\sigma_1, 1024\Sigma, 27552\sigma_2, 4536\Theta\Theta_2, 5664\Psi(\Theta\sigma)_1, 25376\mu\mu_1, 3328\Delta\sigma_1, 21984\Delta_1, 7024\Theta\Lambda_1, 3690\Theta(\Theta\sigma)_1, 3888\Psi\sigma_1, 8384\Lambda\mu_1, 3812\mu\Theta_2, 13120(\mu\sigma)_2, 4928\Theta\sigma_2, 2512\Gamma\Theta_1, 8400\Theta_1\Theta_2, 4896\Delta\mu_1, 1936\Lambda\Theta_2, 8336\mu\Lambda_1, 5440\mu(\Theta\sigma)_1, 3968\Theta\Delta_1, 2304\sigma_1\Theta_2, 1960\Gamma\sigma_1, 15584\Psi_1, 4048\Psi\mu_1, 4736\mu\sigma_2, 2104\Lambda(\Theta\sigma)_1, 1880\Delta\Theta_2, 2208\Lambda\Lambda_1, 4456\Theta_1\sigma_2, 400\Sigma\Theta_1, 2552\Theta(\mu\sigma)_1, 5488\mu_2, 3776(\Lambda\sigma)_1, 1832\Delta\Lambda_1, 2832\mu\Delta_1, 1080\Delta(\Theta\sigma)_1, 368\Psi\Theta_2, 1272\Lambda\sigma_2, 360\Sigma\sigma_1, 1456\Gamma\mu_1, 296\Theta\sigma_1\Theta_2, 1224\sigma_1\sigma_2, 736\mu_1\Theta_2, 2616\Theta\Psi_1, 2096\mu(\mu\sigma)_1, 1412\Psi\Lambda_1, 984\Psi(\Theta\sigma)_1, 1192\Lambda\Delta_1, 948\Delta\sigma_2, 2636\Gamma_1, 848\Theta\mu_2, 1416(\Delta\sigma)_1, 88\Gamma\Theta_2, 688\Theta\Theta_1\sigma_2, 678\Theta(\Lambda\sigma)_1, 352\Delta\Delta_1, 708\Lambda(\mu\sigma)_1, 216\mu\sigma_1\Theta_2, 1460\mu\Psi_1, 605\Psi\sigma_2, 464\Lambda_1\Theta_2, 192\Sigma\mu_1, 116\Theta\mu_1\sigma_2, 352\Gamma(\Theta\sigma)_1, 488\Gamma\Lambda_1, 152\Theta\sigma_1\sigma_2, 448\mu(\Lambda\sigma)_1, 280\mu\mu_2, 164\Lambda\sigma_1\Theta_2, 448\mu\Theta_1\sigma_2, 512\Lambda\Psi_1, 208\Psi\Delta_1, 352\Delta(\mu\sigma)_1, 144\Theta_1\mu_2, 168\Gamma\sigma_2, 120(\Theta\sigma)_2, 172\Theta(\Delta\sigma)_1, 192\Lambda_2, 376\Theta\Gamma_1, 516(\Psi\sigma)_1, 136\Lambda\Theta_1\sigma_2, 168\Delta\Psi_1, 80\Lambda(\Delta\sigma)_1, 72\mu\sigma_1\sigma_2, 216\Psi(\mu\sigma)_1, 80\Lambda\mu_2, 64\Lambda_1\sigma_2, 60(\Theta\sigma)_1\sigma_2, 128\sigma_1\mu_2, 40\Sigma(\Theta\sigma)_1, 152\Theta\mu_1\sigma_2, 64\Sigma\Lambda_1, 40\Gamma\Delta_1, 56\Delta\Theta_1\sigma_2, 80\Psi\Psi_1, 64\mu(\Delta\sigma)_1, 24\Delta\mu_2, 80\Delta(\Lambda\sigma)_1, 112\mu\Gamma_1, 8\Lambda\sigma_1\sigma_2, 60\Sigma_1, 56\Theta(\Psi\sigma)_1, 52\Gamma(\mu\sigma)_1, 36\Theta\Lambda_2, 16\Sigma\sigma_2, 8\Psi\mu_2, 24\Lambda\Gamma_1, 8\Delta\sigma_1\sigma_2, 32\Lambda(\Delta\sigma)_1, 36\mu\mu_1\sigma_2, 20\Psi\Theta_1\sigma_2, 30\Psi(\Lambda\sigma)_1, 24(\Gamma\sigma)_1, 4\Theta(\Theta\sigma)_1\sigma_2, 8\Theta\Lambda_1\sigma_2, 20\Delta_1\sigma_2, 12\Gamma\Psi_1, 12\Theta\sigma_1\mu_2, 8\mu\Lambda_2, 4\Lambda\mu_1\sigma_2, 8\Delta\Gamma_1, 12\mu(\Psi\sigma)_1, 4\Sigma(\mu\sigma)_1, 2\Gamma\Theta_1\sigma_2, 3\Gamma(\Lambda\sigma)_1, 4\Theta_1\Lambda_2, 2\Lambda(\Psi\sigma)_1, 2\Delta\mu_1\sigma_2, 2\Psi\Gamma_1, \Theta(\Gamma\sigma)_1$

TABLE 2-3 (CONTINUED)

product	composition factors
$S \otimes S$	$703872k, 181896\Theta, 203296\mu, 275136\Theta_1, 65792\Lambda, 130360\sigma_1, 41056\Delta,$ $65904\Theta\Theta_1, 43488\Psi, 483616\mu_1, 32864\Theta\sigma_1, 90256\mu\Theta_1, 126960\Theta_2,$ $19856\Gamma, 47768\mu\sigma_1, 29616\Lambda\Theta_1, 99968\Theta\mu_1, 58960(\Theta\sigma)_1, 133408\Lambda_1,$ $18720\Delta\Theta_1, 13928\Lambda\sigma_1, 3320\Sigma, 125560\sigma_2, 23408\Theta\Theta_2, 20016\Psi\Theta_1,$ $108160\mu\mu_1, 10360\Delta\sigma_1, 101760\Delta_1, 28096\Theta\Lambda_1, 14136\Theta(\Theta\sigma)_1, 11668\Psi\sigma_1,$ $35840\Lambda\mu_1, 21480\mu\Theta_2, 56800(\mu\sigma)_1, 23200\Theta\sigma_2, 9080\Gamma\Theta_1, 8336\Theta_1\Theta_2,$ $20864\Delta\mu_1, 9264\Lambda\Theta_2, 32736\mu\Lambda_1, 20348\mu(\Theta\sigma)_1, 19296\Theta\Delta_1, 12800\sigma_1\Theta_2,$ $5664\Gamma\sigma_1, 81056\Psi_1, 17920\Psi\mu_1, 22032\mu\sigma_2, 7136\Lambda(\Theta\sigma)_1, 3304\Delta\Theta_2,$ $9408\Lambda\Lambda_1, 22376\Theta_1\sigma_2, 1488\Sigma\Theta_1, 11232\Theta(\mu\sigma)_1, 33440\mu_2, 14304(\Lambda\sigma)_1,$ $840\Theta\Theta_1\Theta_2, 6912\Delta\Lambda_1, 16480\mu\Delta_1, 4224\Delta(\Theta\sigma)_1, 2784\Psi\Theta_2, 6320\Lambda\sigma_2,$ $1008\Sigma\sigma_1, 6720\Gamma\mu_1, 2064\Theta\sigma_1\Theta_2, 7032\sigma_1\sigma_2, 7520\mu_1\Theta_2, 14224\Theta\Psi_1,$ $480\mu\Theta_1\Theta_2, 10592\mu(\mu\sigma)_1, 5696\Psi\Lambda_1, 4160\Psi(\Theta\sigma)_1, 6208\Lambda\Delta_1, 4272\Delta\sigma_2,$ $17296\Gamma_1, 5376\Theta\mu_2, 8304(\Delta\sigma)_1, 912\Gamma\Theta_2, 3728\Theta\Theta_1\sigma_2, 2888\Theta(\Lambda\sigma)_1,$ $2624\Delta\Delta_1, 3680\Lambda(\mu\sigma)_1, 1944\mu\sigma_1\Theta_2, 9696\mu\Psi_1, 2960\Psi\sigma_2, 368\Lambda\Theta_1\Theta_2,$ $5168\Lambda_1\Theta_2, 928\Sigma\mu_1, 8096\mu_1\sigma_2, 1704\Gamma(\Theta\sigma)_1, 2112\Gamma\Lambda_1, 1056\Theta\sigma_1\sigma_2,$ $296(\Theta\sigma)_1\Theta_2, 736\Theta\mu_1\Theta_2, 2304\mu(\Lambda\sigma)_1, 2576\mu\mu_2, 1096\Lambda\sigma_1\Theta_2,$ $2868\mu\Theta_1\sigma_2, 3296\Lambda\Psi_1, 1824\Psi\Delta_1, 1888\Delta(\mu\sigma)_1, 1808\Theta_1\mu_2, 912\Gamma\sigma_2,$ $1448(\Theta\sigma)_2, 1320\Theta(\Delta\sigma)_1, 1560\Lambda_2, 88\Sigma\Theta_2, 2616\Theta\Gamma_1, 3836(\Psi\sigma)_1,$ $848\Lambda\Theta_1\sigma_2, 1408\Delta\Psi_1, 528\Lambda(\Lambda\sigma)_1, 216\Delta\sigma_1\Theta_2, 600\mu\sigma_1\sigma_2, 1360\Psi(\mu\sigma)_1,$ $352\mu\mu_1\Theta_2, 864\Lambda\mu_2, 612\Lambda_1\sigma_2, 464\Theta\Lambda_1\Theta_2, 572(\Theta\sigma)_1\sigma_2, 1168\sigma_1\mu_2,$ $244\Sigma(\Theta\sigma)_1, 1152\Theta\mu_1\sigma_2, 296\Sigma\Lambda_1, 496\Gamma\Delta_1, 164\Psi\sigma_1\Theta_2, 208\Lambda\mu_1\Theta_2,$ $464\Delta\Theta_1\sigma_2, 818\Psi\Psi_1, 728\mu(\Delta\sigma)_1, 272\Delta\mu_2, 448\Delta(\Lambda\sigma)_1, 1120\mu\Gamma_1,$ $104\Lambda\sigma_1\sigma_2, 784\Sigma_1, 176(\mu\sigma)_1\Theta_2, 528\Theta(\Psi\sigma)_1, 400\Gamma(\mu\sigma)_1, 144\Theta\Theta_1\mu_2,$ $120\Theta(\Theta\sigma)_2, 264\Theta\Lambda_2, 104\Sigma\sigma_2, 40\sigma_3, 128\Psi\mu_2, 320\Lambda\Gamma_1, 188\mu\Lambda_1\Theta_2,$ $88\Delta\sigma_1\sigma_2, 304\Lambda(\Delta\sigma)_1, 440\mu\mu_1\sigma_2, 232\Psi\Theta_1\sigma_2, 224\Psi(\Lambda\sigma)_1,$ $316(\Gamma\sigma)_1, 52\Theta(\Theta\sigma)_1\sigma_2, 76\Theta\Lambda_1\sigma_2, 348\Delta_1\sigma_2, 48\mu_1\mu_2, 184\Gamma\Psi_1,$ $40\Sigma\Delta_1, 40\Gamma\sigma_1\Theta_2, 128\Theta\sigma_1\mu_2, 20\Psi\sigma_1\sigma_2, 68\mu\Lambda_2, 56\mu(\Theta\sigma)_2,$ $88\Lambda\mu_1\sigma_2, 80\Lambda\Lambda_1\Theta_2, 24\mu\Theta_1\mu_2, 112\Delta\Gamma_1, 180\mu(\Psi\sigma)_1, 64\Delta(\Delta\sigma)_1,$ $40\Sigma(\mu\sigma)_1, 32(\mu\sigma)_1\sigma_2, 60\Theta\Sigma_1, 18\Gamma\mu_2, 44\Gamma\Theta_1\sigma_2, 44\Gamma(\Lambda\sigma)_1, 72\Theta_1\Lambda_2,$ $56(\Lambda\sigma)_1\Theta_2, 8\Lambda\Lambda_2, 20\Lambda(\Theta\sigma)_2, 14\mu\Lambda_1\sigma_2, 14\mu(\Theta\sigma)_1\sigma_2, 8\Lambda\Theta_1\mu_2,$ $24\mu\sigma_1\mu_2, 44\Lambda(\Psi\sigma)_1, 32\Psi(\Delta\sigma)_1, 40\Delta\mu_1\sigma_2, 40\Psi\Gamma_1, 16\Lambda_1\mu_2, 20\Theta\Delta_1\sigma_2,$ $12\Sigma\Psi_1, 4\sigma_1\Lambda_2, 28\Theta(\Gamma\sigma)_1, 18\Psi_1\sigma_2, 12(\Theta\sigma)_1\mu_2, 4\sigma_1(\Theta\sigma)_2, 8\mu\Sigma_1,$ $8\Lambda\sigma_1\mu_2, 8\Psi\mu_1\sigma_2, 12\Delta(\Psi\sigma)_1, 4\Delta\Lambda_2, 4\Theta\Theta_1\Lambda_2, 2\Sigma(\Lambda\sigma)_1, 4\Gamma(\Delta\sigma)_1,$ $4(\mu\sigma)_2, 4\Gamma\Gamma_1, 2\Sigma\Theta_1\sigma_2, 2\Psi(\Psi\sigma)_1, 2\mu\Delta_1\sigma_2, 2\Lambda\Sigma_1, 2\mu(\Gamma\sigma)_1, S_1$

CHAPTER 3

INFINITESIMAL SUBGROUP SCHEMES

§3.1 Lyndon-Hochschild-Serre Spectral Sequence

We use the notation established in the introduction, denoting by G and $\tilde{G}$ the simply connected groups of type B_4 and C_4 respectively. When considering the simply connected group of type A_4 alone, denote it also by G. Denote by G_1 (resp. $\tilde{G}_1$, resp. D_1, etc.) the first Frobenius kernel of the algebraic group G (resp. $\tilde{G}$, resp. D, etc.). For the algebraic groups of type B_4 and C_4, denote by G_σ and $\tilde{G}_\tau$ the kernels of the maps $\sigma : G \to \tilde{G}$ and $\tau : \tilde{G} \to G$ defined as follows [21, 22]

$$\sigma : x_\alpha(t) \mapsto \begin{cases} \tilde{x}_{\alpha^\vee}(t) & \text{if } \alpha \text{ is a long root} \\ \tilde{x}_{\alpha^\vee}(t^2) & \text{if } \alpha \text{ is a short root} \end{cases} \tag{3-1}$$

and

$$\tau : \tilde{x}_\alpha(t) \mapsto \begin{cases} x_{\alpha^\vee}(t) & \text{if } \alpha \text{ is a long root} \\ x_{\alpha^\vee}(t^2) & \text{if } \alpha \text{ is a short root} \end{cases} \tag{3-2}$$

The tangent maps are

$$d\sigma : X_\alpha \mapsto \begin{cases} \tilde{X}_{\alpha^\vee} & \text{if } \alpha \text{ is a long root} \\ 0 & \text{if } \alpha \text{ is a short root} \end{cases} \tag{3-3}$$

and

$$d\tau : \tilde{X}_\alpha \mapsto \begin{cases} X_{\alpha^\vee} & \text{if } \alpha \text{ is a long root} \\ 0 & \text{if } \alpha \text{ is a short root} \end{cases} \tag{3-4}$$

We now compute the modules $\mathrm{Ext}^1_{G_1}(L, M)$ (resp. $\mathrm{Ext}^1_{D_1}(L, M)$) for each pair of restricted simple modules for the simply connected algebraic groups of types A_4 (resp. D_4), and then proceed to compute the G- and $\bar{G}$- modules $\mathrm{Ext}^1_{G_\sigma}(L, M)$ and $\mathrm{Ext}^1_{\bar{G}_\tau}(L, M)$ for the simply connected algebraic groups of types B_4 and C_4. The interpretation of $G_1 := Ker(F)$ (F being the Frobenius morphism) as an infinitesimal normal subgroup with $G \cong G/G_1$ was developed by Demazure and Gabriel [8] as part of the foundations of the subject of algebraic groups. The Lyndon-Hochschild-Serre spectral sequence for the pair (G, G_1) (similarly for (D, D_1)) is

$$\mathrm{Ext}^i_G(L(\bar{\lambda}), \mathrm{Ext}^j_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\bar{\mu})) \Rightarrow_i \mathrm{Ext}^{i+j}_G(L(\lambda), L(\mu)),$$

where we write $\lambda = \lambda^0 + 2\bar{\lambda}$ with $\lambda^0, \bar{\lambda} \in X_1(T)$ (and similarly for μ.) The resulting 5-term exact sequence is

$$0 \to \mathrm{Ext}^1_G(L(\bar{\lambda}), \mathrm{Hom}_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\bar{\mu})) \to \mathrm{Ext}^1_G(L(\lambda), L(\mu))$$

$$\to \mathrm{Hom}_G(L(\bar{\lambda}), \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\bar{\mu}))$$

$$\to \mathrm{Ext}^2_G(L(\bar{\lambda}), \mathrm{Hom}_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\bar{\mu})) \to \mathrm{Ext}^2_G(L(\lambda), L(\mu))$$

Later, we shall use the 5-term sequence that results from the Lyndon-Hochschild-Serre spectral sequence for the pair (G, G_σ)

$$0 \to \mathrm{Ext}^1_{\bar{G}}(\bar{L}(\bar{\lambda}), \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \bar{L}(\bar{\mu})) \to \mathrm{Ext}^1_G(L(\lambda), L(\mu))$$

$$\to \mathrm{Hom}_{\bar{G}}(\bar{L}(\bar{\lambda}), \mathrm{Ext}^1_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \bar{L}(\bar{\mu}))$$

$$\to \mathrm{Ext}^2_{\bar{G}}(\bar{L}(\bar{\lambda}), \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \bar{L}(\mu)) \to \mathrm{Ext}^2_G(L(\lambda), L(\mu))$$

We will also use the 5-term sequence for the pair (G_1, G_σ) to calculate the modules $\mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))$ for 2-restricted λ, μ

$$0 \to \mathrm{Ext}^1_{\tilde{G}_r}(\tilde{L}(\bar{\lambda}), \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \tilde{L}(\bar{\mu}))^{(\sigma)} \to \mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))$$
$$\to \mathrm{Hom}_{\tilde{G}_r}(\tilde{L}(\bar{\lambda}), \mathrm{Ext}^1_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \tilde{L}(\bar{\mu}))^{(\sigma)}$$
$$\to \mathrm{Ext}^2_{\tilde{G}_r}(\tilde{L}(\bar{\lambda}), \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \tilde{L}(\bar{\mu}))^{(\sigma)} \to \mathrm{Ext}^2_{G_1}(L(\lambda), L(\mu))$$

Similarly, we can write down 5-term sequences for the pairs $(\tilde{G}, \tilde{G}_r)$ and $(\tilde{G}_1, \tilde{G}_r)$ (Sequences of this type were used by Donkin [9] to obtain some results about algebraic group cohomology.) It is known that $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\lambda^0)) = 0$ for restricted weights λ^0. If $\lambda^0 \neq \mu^0$, we have from the 5-term sequence that

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(L(\lambda), \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\bar{\mu}))$$

We shall use this isomorphism together with information obtained from Weyl modules about $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$ for various choices of λ, μ, to deduce the structure of the modules $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$. The strategy is to first test whether certain modules appear in the socle of $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})}$ by tensoring with various modules of the form $L(\mu)$, and using the above isomorphism to determine which modules might appear in the socle of the resulting tensor product. Once the socle of $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})}$ is determined, we deduce whether various modules L appear in the next socle layer by tensoring with modules of the form $L(\mu)$ for choices of μ with the property that there are modules in the socle of $L \otimes L(\mu)$ that have no non-split extensions by M for all composition factors of $L(\bar{\mu}) \otimes \mathrm{soc}(\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})})$. The search for composition factors of

$$\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$$

is considerably narrowed by

Lemma 3.1.1 *Let $\lambda^0, \mu^0 \in X_1(T)$. If $L(\bar{\lambda})$ appears as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})}$, then $L(\lambda^0 + 2\bar{\lambda})$ must appear as a composition factor of $L(\rho) \otimes L(\rho - \mu^0)^*$, where ρ is the halfsum of the positive roots.*

Proof Since $L(\rho)$ is injective for G_1, then so is $L(\rho) \otimes L(\rho - \mu^0)^*$, and the multiplicity of $Q(\mu^0)$ as a direct summand is $\dim_k(\mathrm{Hom}_{G_1}(L(\mu^0), L(\rho) \otimes L(\rho - \mu^0)^*)) = \dim_k(\mathrm{Hom}_{G_1}(L(\mu^0) \otimes L(\rho - \mu^0), L(\rho))) = 1$. Thus,

$$\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0)) \cong \mathrm{Hom}_{G_1}(L(\lambda^0), [L(\rho) \otimes L(\rho - \mu^0)^*]/\mathrm{soc}[L(\rho) \otimes L(\rho - \mu^0)^*]).$$

Now, for any $\lambda \in X_1(T)$ and any G module M, the evaluation map

$$\mathrm{Hom}_{G_1}(L(\lambda), M) \otimes L(\lambda) \to M$$

is injective, since any simple submodule of the tensor product must be of the form $L(2\bar{\lambda}) \otimes L(\lambda)$ where $L(2\bar{\lambda})$ is a simple submodule of $\mathrm{Hom}_{G_1}(L(\lambda), M)$, and thus could not lie in the kernel of the evaluation map. This is because

$$\mathrm{Hom}_G(L(2\bar{\lambda}) \otimes L(\lambda), L_1 \otimes L(\lambda)) \cong [(L(2\bar{\lambda}) \otimes L(\lambda))^* \otimes L_1 \otimes L(\lambda)]^G$$

$$\cong \{[(L(2\bar{\lambda}) \otimes L(\lambda))^* \otimes L_1 \otimes L(\lambda)]^{G_1}\}^{G/G_1} \cong \{(L(2\bar{\lambda}))^* \otimes L_1 \otimes [L(\lambda)^* \otimes L(\lambda)]^{G_1}\}^{G/G_1}$$

$$\cong \{(L(2\bar{\lambda}))^* \otimes L_1 \otimes k\}^{G/G_1} \cong \mathrm{Hom}_G(L(2\bar{\lambda}), L_1)$$

for any $\bar{\lambda}, \lambda \in X_1(T)$ and any G-module L. Therefore, for each composition factor $L(2\bar{\lambda})$ of $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$, we must have a corresponding composition factor

$L(2\tilde{\lambda}) \otimes L(\lambda^0)$ of $[L(\rho) \otimes L(\rho - \mu^0)^*]/\mathrm{soc}[L(\rho) \otimes L(\rho - \mu^0)^*]$, and thus of $[L(\rho) \otimes L(\rho - \mu^0)^*]$

In order to proceed with the strategy of searching for composition factors of $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$ by tensoring with other modules, it is necessary to compute the socles of certain tensor products of simple restricted modules. We also need this information for the final computation of $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$. In the next lemma, we show that $\mathrm{Hom}_{G_1}(L(\lambda^0), L(\beta^0) \otimes L(\mu^0))$ is G-trivial in many cases if β^0, λ^0, and μ^0 are 2-restricted. This will simplify immensely the final computation of the Ext^1_Gs.

Lemma 3.1.2. *Let $\beta^0 \in X_1(T)$ be a restricted weight with the property that*

$$\langle \beta^0, \rho \rangle < \min_{\omega \in X_1(T) \setminus \{0\}} \langle 2\omega, \rho \rangle$$

Let $\lambda^0, \mu^0 \in X_1(T)$. Then

$$Hom_{G_1}(L(\lambda^0), L(\beta^0) \otimes L(\mu^0))$$

is G-trivial.

Proof. By taking duals if necessary, assume that $\langle \mu^0, \rho \rangle \leq \langle \lambda^0, \rho \rangle$. Let $L(2\delta)$ be a nontrivial composition factor of $\mathrm{Hom}_{G_1}(L(\lambda^0), L(\beta^0) \otimes L(\mu^0))$. Since the evaluation map

$$\mathrm{Hom}_{G_1}(L(\lambda^0), L(\beta^0) \otimes L(\mu^0)) \otimes L(\lambda^0) \to L(\beta^0) \otimes L(\mu^0),$$

is injective, $L(\lambda^0) \otimes L(2\delta)$ must be a composition factor of $L(\beta^0) \otimes L(\mu^0)$. Then we have $\langle \lambda^0 + 2\delta, \rho \rangle \leq \langle \beta^0 + \mu^0, \rho \rangle$, a contradiction.

Next, we compute the G_1 socles of those tensor products of restricted simple modules that are needed in our ext computations. Because of the limited number

of isomorphism types of simple modules which appear as composition factors of the $\mathrm{Ext}^1_{G_1}s$ (resp., $\mathrm{Ext}^1_{D_1}s$, $\mathrm{Ext}^1_{G_s}s$, $\mathrm{Ext}^1_{\tilde{G}_r}s$,) it turns out that we are always able to apply Lemma 3.1.2. In particular, we may apply the lemma when $\beta^0 = \lambda_1$ or λ_2 for A_4 and when $\beta^0 = \delta_1$ or δ_2 for D_4. Therefore, in the next two sections we will use the lemma freely without constantly referring to it by number.

In this chapter and the next we occasionally quote some results computed by Sin [20] about various quantities of the form $\mathrm{Ext}^i_{D_1}(L, M)$. In Sin's work [20, Lemmas 4.5, 4.6 and 4.9], these quantities are actually computed for a quotient $\tilde{D}_1$ of D_1 by a subgroup scheme of multiplicative type, but $H^*(D_1, \ ) \cong H^*(\tilde{D}_1, \ )$ (cf. Steinberg [21].)

§3.2 Socles of Tensor Products for A_4

We now use the lemmas of the preceding section to compute the G_1-socles of the tensor products of either $L(\lambda_1)$ or $L(\lambda_2)$ with the other restricted simple modules. Using the fact that there are 15 non-trivial restricted simple modules, and exploiting symmetry (e.g. we do not have to compute $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_1))$ or $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_4))$ over again), we now list $15 + 13 = 28$ such computations. For notational convenience we adopt the following convention for displaying filtrations of a module. The subquotients will be listed from top to bottom, seperated by the symbol "//".

a) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_1)) \cong L(\lambda_2)$

By Table 2.2, the only simple restricted G module types that appear as G- composition factors is $L(\lambda_2)$. We have

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_1)) \cong k,$$

by considering a good filtration of

$L(\lambda_1) \otimes L(\lambda_1) \cong H^0(\lambda_1) \otimes H^0(\lambda_1) \cong H^0(2\lambda_1)//H^0(\lambda_2)$.

b) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_4)) \cong k \oplus L(\lambda_1 + \lambda_4)$

By Table 2-2, the only simple restricted module types that appear as G- composition factors are k and $L(\lambda_1 + \lambda_4)$. We have

$$\mathrm{Hom}_G(k, L(\lambda_1) \otimes L(\lambda_4)) \cong \mathrm{Hom}_G(L(\lambda_4), L(\lambda_1)) \cong k,$$

by Schur's lemma, and

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_4), L(\lambda_1) \otimes L(\lambda_4)) \cong k,$$

by Table 2-2, and the fact that $L(\lambda_1 + \lambda_4)$ is in a different linkage class than k.

c) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_2)) \cong L(\lambda_3) \oplus L(\lambda_1 + \lambda_2)$.

This is immediate from Table 2-2 as we have

$$\mathrm{Ext}^1_G(L(\lambda_3), L(\lambda_1 + \lambda_2)) \cong 0,$$

(as λ_3 and $\lambda_1 + \lambda_2$ lie in different linkage classes.)

d) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_3)) \cong L(\lambda_4)$.

The only simple restricted module types that appear as G- composition factors are $L(\lambda_4), L(\lambda_1 + \lambda_3)$. We have

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_3), L(\lambda_1) \otimes L(\lambda_3)) \cong \mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_2 + \lambda_4)) \cong 0,$$

by Table 2-2,

$$\mathrm{Hom}_G(L(\lambda_4), L(\lambda_1) \otimes L(\lambda_3)) \cong k,$$

by considering a good filtration of $L(\lambda_1) \otimes L(\lambda_3) \cong H^0(\lambda_1) \otimes H^0(\lambda_3) \cong H^0(\lambda_1 + \lambda_3)//H^0(\lambda_4)$

e) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_1 + \lambda_4)) \cong L(\lambda_1)$.

By Table 2-2, the only simple restricted module types that appear as G composition factors are $L(\lambda_1)$ and $L(\lambda_2 + \lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_1), L(\lambda_1) \otimes L(\lambda_1 + \lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1 + \lambda_4), L(\lambda_1) \otimes L(\lambda_4)) \cong k,$$

by 3.2(b) above, and

$$\mathrm{Hom}_G(L(\lambda_2 + \lambda_4), L(\lambda_1) \otimes L(\lambda_1 + \lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1 + \lambda_4), L(\lambda_1) \otimes L(\lambda_1 + \lambda_3)) \cong 0,$$

by Table 2-2.

f) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_2 + \lambda_4)) \cong L(\lambda_3 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_4)$

This is immediate from Table 2-2 as we have

$$\mathrm{Ext}^1_G(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong 0,$$

(as $\lambda_3 + \lambda_4$ and $\lambda_1 + \lambda_2 + \lambda_4$ lie in different linkage classes.)

g) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_1 + \lambda_3)) \cong L(\lambda_2 + \lambda_3)$

The only simple restricted G module types that appear as G- composition factors are k and $L(\lambda_2 + \lambda_3)$. We have

$$\mathrm{Hom}_G(k, L(\lambda_1) \otimes L(\lambda_1 + \lambda_3)) \cong \mathrm{Hom}_G(L(\lambda_4), L(\lambda_1 + \lambda_3)) \cong 0,$$

by Schur's lemma, and

$$\mathrm{Hom}_G(L(\lambda_2 + \lambda_3), L(\lambda_1) \otimes L(\lambda_1 + \lambda_3)) \cong k,$$

by considering a good filtration and decomposition into linkage classes of

$$H^0(\lambda_1) \otimes H^0(\lambda_1 + \lambda_3) \cong L(\lambda_1) \otimes L(\lambda_4) // L(\lambda_1) \otimes L(\lambda_1 + \lambda_3)$$

$$\cong H^0(\lambda_1 + \lambda_4) \oplus \{H^0(2\lambda_1 + \lambda_3) // H^0(\lambda_2 + \lambda_3)\},$$

and by observing that

$$\mathrm{Hom}_G(L(\lambda_2 + \lambda_3), L(\lambda_1) \otimes L(\lambda_4)) \cong 0,$$

by Table 2-2.

h) $\mathrm{Soc}_{G_1}(L(\lambda_1) \otimes L(\lambda_3 + \lambda_4)) \cong L(\lambda_3)$

The only simple restricted G-module types that appear as G- composition factors are $L(\lambda_3)$ and $L(\lambda_1 + \lambda_3 + \lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_3 + \lambda_4), L(\lambda_1) \otimes L(\lambda_3 + \lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_4)) \cong 0,$$

by Table 2-2, and

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1)\otimes L(\lambda_3+\lambda_4)) \cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_1)\otimes L(\lambda_2)) \cong k,$$

by 3.2(c) above.

i) $\mathrm{Soc}_{G_1}(L(\lambda_1)\otimes L(\lambda_1+\lambda_2)) \cong L(\lambda_1+\lambda_3)$.

By Table 2-2, the simple restricted module types that appear as G- composition factors are $L(\lambda_4)$ and $L(\lambda_1+\lambda_3)$. We have

$$\mathrm{Hom}_G(L(\lambda_4), L(\lambda_1)\otimes L(\lambda_1+\lambda_2)) \cong \mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_1)) \cong 0$$

by Table 2-2, and

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_3), L(\lambda_1)\otimes L(\lambda_1+\lambda_2))$$

$$\cong \mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_2+\lambda_4)) \cong k$$

by 3.2(f) above.

j) $\mathrm{Soc}_{G_1}(L(\lambda_1)\otimes L(\lambda_2+\lambda_3)) \cong L(\lambda_2+\lambda_4) \oplus L(\lambda_1+\lambda_2+\lambda_3)$.

The simple restricted module types that appear as G- composition factors are $L(\lambda_2+\lambda_4)$ and $L(\lambda_1+\lambda_2+\lambda_3)$. We have

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_4), L(\lambda_1)\otimes L(\lambda_2+\lambda_3))$$

$$\cong \mathrm{Hom}_G(L(\lambda_2+\lambda_3), L(\lambda_1)\otimes L(\lambda_1+\lambda_3)) \cong k,$$

by 3.2(g) above, and

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_3+\lambda_3), L(\lambda_1)\otimes L(\lambda_2+\lambda_3)) \cong k,$$

by Table 2.2, and the fact that $L(\lambda_1+\lambda_2+\lambda_3)$ lies in a different linkage class than the other composition factors of $L(\lambda_1)\otimes L(\lambda_2+\lambda_3)$.

k) $\mathrm{Soc}_{G_1}(L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_4)) \cong L(\lambda_1+\lambda_3+\lambda_4)$

The simple restricted module types that appear as G- composition factors are $L(\lambda_3)$ and $L(\lambda_1+\lambda_3+\lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_3), L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_2)) \cong 0,$$

by Table 2.2, and so, we must have

$$0 \neq \mathrm{Hom}_G(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_4)),$$

$$\preceq \mathrm{Hom}_G(V(\lambda_1+\lambda_3+\lambda_4), H^0(\lambda_1)\otimes H^0(\lambda_1+\lambda_2+\lambda_4)) \cong k,$$

by considering a good filtration of

$H^0(\lambda_1)\otimes H^0(\lambda_1+\lambda_2+\lambda_4)) \cong H^0(2\lambda_1+\lambda_2+\lambda_4)//H^0(2\lambda_2+\lambda_4)//H^0(\lambda_1+\lambda_3+\lambda_4)//H^0(\lambda_1+\lambda_2)$.

l) $\mathrm{Soc}_{G_1}(L(\lambda_1)\otimes L(\lambda_2+\lambda_3+\lambda_4)) \cong L(\lambda_1+\lambda_3)\oplus L(\lambda_2+\lambda_3+\lambda_4)$

The simple restricted module types that appear as G- composition factors are $L(\lambda_1+\lambda_3)$ and $L(\lambda_2+\lambda_3+\lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_3), L(\lambda_1)\otimes L(\lambda_1+\lambda_3+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_1)\otimes L(\lambda_2+\lambda_4)) \cong k,$$

by 3.2(g), and

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_1+\lambda_3+\lambda_4)) \cong k,$$

by considering a good filtration and decomposition into linkage classes of

$$H^0(\lambda_1)\otimes H^0(\lambda_1+\lambda_3+\lambda_4) \cong L(\lambda_1)\otimes L(\lambda_2)//L(\lambda_1)\otimes L(2\lambda_4)//L(\lambda_1)\otimes L(\lambda_1+\lambda_3+\lambda_4)$$

$$\cong \{H^0(\lambda_1+2\lambda_4)//H^0(\lambda_1+\lambda_2)\} \oplus \{H^0(2\lambda_1+\lambda_3+\lambda_4)//H^0(\lambda_2+\lambda_3+\lambda_4)\},$$

and by observing that

$$\mathrm{Hom}_G((L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_3)) \cong 0,$$

by Table 2.2

m) $\mathrm{Soc}_{G_1}(L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_3)) \cong L(\lambda_1+\lambda_2+\lambda_4)$

The simple restricted module types that appear as G composition factors are $L(\lambda_2)$ and $L(\lambda_1+\lambda_2+\lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_3))$$

$$\cong \mathrm{Hom}_G(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_3)) \cong 0,$$

$$\mathrm{Hom}_G(L(\lambda_3), L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_3))$$

$$\cong \mathrm{Hom}_G(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_2)) \cong 0,$$

and

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_3))$$

$$\cong \mathrm{Hom}_G(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1)\otimes L(\lambda_1+\lambda_2+\lambda_3)) \cong k,$$

by 3.2(i) above.

n) $\mathrm{Soc}_{G_1}(L(\lambda_1)\otimes L(\lambda_2+\lambda_3+\lambda_4)) \cong L(\lambda_2+\lambda_3)\oplus L(\rho)$.

The simple restricted module types that appear as G composition factors are k, $L(\lambda_2+\lambda_3)$ and $L(\rho)$. We have

$$\mathrm{Hom}_G(k, L(\lambda_1)\otimes L(\lambda_2+\lambda_3+\lambda_4)) \cong 0,$$

by Schur's lemma. Also, we have

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_3), L(\lambda_1)\otimes L(\lambda_2+\lambda_3+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_3), L(\lambda_1)\otimes L(\lambda_2+\lambda_3)) \cong k,$$

by 3.2(j) above. Finally,

$$\mathrm{Hom}_{G_1}(L(\rho), L(\lambda_1)\otimes L(\lambda_2+\lambda_3+\lambda_4)) \cong k,$$

by Table 2-2, and the fact that $L(\rho)$ is injective for G_1.

o) $\mathrm{Soc}_{G_1}(L(\lambda_1)\otimes L(\rho)) \cong L(\lambda_1+\lambda_2+\lambda_3)$

We have

$$\mathrm{Hom}_G(L(\nu^0), L(\lambda_1)\otimes L(\rho)) \cong \mathrm{Hom}_G(L(\rho), L(\lambda_1)\otimes L(\nu^0)^*) \cong 0,$$

for all restricted weights ν^0 except $\nu^0 = \lambda_1 + \lambda_2 + \lambda_3$ by Table 2.2, and

$$\mathrm{Hom}_G(L(\rho), L(\lambda_1) \otimes L(\lambda_1 + \lambda_2 + \lambda_3)^*) \cong k$$

by 3.2(n) above.

p) $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_2)) \cong L(\lambda_4) \oplus L(\lambda_1 + \lambda_3)$

The only simple restricted module types that appear as G- composition factors are $L(\lambda_4)$ and $L(\lambda_1 + \lambda_3)$. We have

$$\mathrm{Hom}_G(L(\lambda_4), L(\lambda_2) \otimes L(\lambda_2))$$

$$\cong \mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k,$$

by 3.2(c) above. Also,

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_3), L(\lambda_2) \otimes L(\lambda_2)) \cong k,$$

by Table 2.2, and by considering a filtration by Weyl modules

$$L(\lambda_2) \otimes L(\lambda_2) \cong V(\lambda_2) \otimes V(\lambda_2) \cong V(\lambda_4) // V(\lambda_1 + \lambda_3) // V(2\lambda_2)$$

together with the structures of $V(\lambda_1 + \lambda_3)$ and $V(2\lambda_2)$ obtained by using Jantzen's sum formula. (cf. Table 1.2)

q) $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_3)) \cong k \oplus L(\lambda_1 + \lambda_4)$

The simple restricted module types that appear as G- composition factors are k, $L(\lambda_1 + \lambda_4)$, and $L(\lambda_2 + \lambda_3)$. We have

$$\mathrm{Hom}_G(k, L(\lambda_2) \otimes L(\lambda_3)) \cong \mathrm{Hom}_G(L(\lambda_2), L(\lambda_3)) \cong k,$$

by Schur's lemma. Also,

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_3), L(\lambda_2)\otimes L(\lambda_3)) \cong 0,$$

by considering a filtration by Weyl modules of

$$L(\lambda_2)\otimes L(\lambda_3) \cong V(\lambda_2)\otimes V(\lambda_3) \cong k//V(\lambda_1+\lambda_4)//V(\lambda_2+\lambda_3)$$

Finally,

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_4), L(\lambda_2)\otimes L(\lambda_3)) \cong k,$$

by Table 2-2 and the fact that $L(\lambda_1+\lambda_4)$ lies in a different linkage class than the other composition factors of $L(\lambda_2)\otimes L(\lambda_3)$.

r) $\mathrm{Soc}_{G_1}(L(\lambda_2)\otimes L(\lambda_1+\lambda_4)) \cong L(\lambda_2)\otimes L(\lambda_3+\lambda_4)$.

The simple restricted G-module types that appear as G- composition factors are $L(\lambda_2), L(\lambda_3+\lambda_4)$ and $L(\lambda_1+\lambda_2+\lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_2)\otimes L(\lambda_1+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_4), L(\lambda_2)\otimes L(\lambda_3)) \cong k,$$

by 3.2(q) above.

$$\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_4)) \cong k,$$

by Table 2-2, and the fact that $L(\lambda_3+\lambda_4)$ lies in a different linkage class than the other composition factors of $L(\lambda_2)\otimes L(\lambda_1+\lambda_4)$. Finally,

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_4)) \cong 0,$$

by considering a filtration by Weyl modules of

$L(\lambda_2) \otimes L(\lambda_1 + \lambda_4) \cong V(\lambda_2) \otimes V(\lambda_1 + \lambda_4) \cong V(\lambda_2) // V(2\lambda_1) // V(\lambda_3 + \lambda_4) // V(\lambda_1 + \lambda_2 + \lambda_4)$,

and the structure of $V(\lambda_1 + \lambda_2 + \lambda_4)$

s) $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_2 + \lambda_4)) \cong L(\lambda_3) \oplus L(\lambda_1 + \lambda_3 + \lambda_4)$.

The simple restricted module types that appear as G- composition factors are $L(\lambda_3)$ and $L(\lambda_1 + \lambda_3 + \lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_2 + \lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1 + \lambda_3), L(\lambda_2) \otimes L(\lambda_2)) \cong k,$$

by 3.2(p) above. We show that

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_3 + \lambda_4), L(\lambda_2) \otimes L(\lambda_2 + \lambda_4) \cong k,$$

by considering a filtration by Weyl modules

$V(\lambda_2) \otimes V(\lambda_2 + \lambda_4) \cong L(\lambda_2) \otimes L(\lambda_2 + \lambda_4) // L(\lambda_2) \otimes L(\lambda_4)$

$\cong V(\lambda_3) // V(2\lambda_4) // V(\lambda_1 + \lambda_2) // V(\lambda_1 + \lambda_3 + \lambda_4) // V(2\lambda_2 + \lambda_4)$

Using the structure of $V(2\lambda_2 + \lambda_4)$ obtained from Jantzen's sum formula, and the observation that

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_3 + \lambda_4), L(\lambda_2) \otimes L(\lambda_4)) \cong 0,$$

(from Table 2-2) we obtain

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_2+\lambda_4)\neq 0$$

However, the corresponding good filtration then gives

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_2+\lambda_4))$$

$$\precsim \mathrm{Hom}_G(V(\lambda_1+\lambda_3+\lambda_4), H^0(\lambda_2)\otimes H^0(\lambda_2+\lambda_4))\cong k.$$

t) $\mathrm{Soc}_{G_1}(L(\lambda_2)\otimes L(\lambda_1+\lambda_3))\cong L(\lambda_2+\lambda_4)\oplus L(\lambda_1+\lambda_2+\lambda_3)$.
The simple restricted module types that appear as G- composition factors are $L(\lambda_1)$, $L(\lambda_2+\lambda_4)$ and $L(\lambda_1+\lambda_2+\lambda_3)$. We have

$$\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2)\otimes L(\lambda_1+\lambda_3))$$

$$\cong \mathrm{Hom}_G(L(\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_4))\cong 0,$$

by 3.2(d) above, and

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_3), L(\lambda_2)\otimes L(\lambda_1+\lambda_3))\cong k,$$

by Table 2-2, and the fact that $L(\lambda_1+\lambda_2+\lambda_3)$ lies in a different linkage class than the other composition factors of $L(\lambda_2)\otimes L(\lambda_1+\lambda_3)$.

Finally, we will show that

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_3))\cong k.$$

From Table 2-2, we obtain

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_3)) \neq 0,$$

by observing that $\mathrm{Hom}_G(L(\nu), L(\lambda_2)\otimes L(\lambda_1+\lambda_3)) \cong 0$, for all composition factors $L(\nu)$ in the same linkage class as $L(\lambda_2+\lambda_4)$. However, by considering a good filtration of

$$H^0(\lambda_2)\otimes H^0(\lambda_1+\lambda_3) \cong H^0(\lambda_1+\lambda_2+\lambda_3)//H^0(2\lambda_1+\lambda_4)//H^0(2\lambda_3)//H^0(\lambda_2+\lambda_4)//H^0(\lambda_1)$$

we obtain

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_3))$$

$$\cong \mathrm{Hom}_G(V(\lambda_2+\lambda_4), H^0(\lambda_2)\otimes H^0(\lambda_1+\lambda_3)) \cong k$$

u) $\mathrm{Soc}_{G_1}(L(\lambda_2)\otimes L(\lambda_3+\lambda_4)) \cong L(\lambda_4)\oplus L(\lambda_2+\lambda_3+\lambda_4)$.

The simple restricted module types that appear as G- composition factors are $L(\lambda_4)$, $L(\lambda_1+\lambda_3)$ and $L(\lambda_2+\lambda_3+\lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_4), L(\lambda_2)\otimes L(\lambda_3+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_2)\otimes L(\lambda_1)) \cong k,$$

by 3.2(c) above, and

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_3), L(\lambda_2)\otimes L(\lambda_3+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_2)\otimes L(\lambda_2+\lambda_4)) \cong 0,$$

by Table 2.2. We also have

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_3+\lambda_4)) \cong k$$

by Table 2.2, and the fact that $L(\lambda_2+\lambda_3+\lambda_4)$ lies in a different linkage class than the other composition factors of $L(\lambda_2)\otimes L(\lambda_3+\lambda_4)$.

v) $\mathrm{Soc}_{G_1}(L(\lambda_2)\otimes L(\lambda_1+\lambda_2)) \cong L(\lambda_2+\lambda_3)\oplus L(\lambda_1+\lambda_4)$

The simple restricted module types that appear as G- composition factors are k, $L(\lambda_2+\lambda_3)$ and $L(\lambda_1+\lambda_4)$. We have

$$\mathrm{Hom}_G(k, L(\lambda_2)\otimes L(\lambda_1+\lambda_2)) \cong 0,$$

by Schur's lemma. We also have

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_2)) \cong k,$$

by Table 2.2, and the fact that $L(\lambda_1+\lambda_4)$ lies in a different linkage class than the other composition factors of $L(\lambda_2)\otimes L(\lambda_1+\lambda_2)$.

Finally,

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_3), L(\lambda_2)\otimes L(\lambda_1+\lambda_2)) \cong k,$$

by considering a good filtration and decomposition into linkage classes of

$H^0(\lambda_2)\otimes H^0(\lambda_1+\lambda_2) \cong L(\lambda_2)\otimes L(\lambda_1+\lambda_2) \cong \{H^0(\lambda_1+2\lambda_2)//H^0(2\lambda_1+\lambda_3)//H^0(\lambda_2+\lambda_3)\}\oplus H^0(\lambda_1+\lambda_4)$.

w) $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_2 + \lambda_3)) \cong L(\lambda_3 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_4)$.

The simple restricted module types that appear as G- composition factors are $L(\lambda_2)$, $L(\lambda_3 + \lambda_4)$, and $L(\lambda_1 + \lambda_2 + \lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_2) \otimes L(\lambda_2 + \lambda_3))$$

$$\cong \mathrm{Hom}_G(L(\lambda_2 + \lambda_3), L(\lambda_2) \otimes L(\lambda_2)) \cong 0,$$

by 3.2(q) above. We also have

$$\mathrm{Hom}_G(L(\lambda_3 + \lambda_4), L(\lambda_2) \otimes L(\lambda_2 + \lambda_3)) \cong k,$$

by Table 2-2, and the fact that $L(\lambda_3 + \lambda_4)$ lies in a different linkage class than the other composition factors of $L(\lambda_2) \otimes L(\lambda_2 + \lambda_3)$. Finally, we claim that

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_2 + \lambda_4), L(\lambda_2) \otimes L(\lambda_2 + \lambda_3)) \cong k$$

To show this, we consider a filtration by Weyl modules of

$$V(\lambda_2) \otimes V(\lambda_2 + \lambda_3) \cong L(\lambda_2) \otimes L(\lambda_2 + \lambda_3) // L(\lambda_2)$$

$$\cong V(\lambda_2) // V(\lambda_3 + \lambda_4) // V(\lambda_1 + \lambda_2 + \lambda_4) // V(\lambda_1 + 2\lambda_3) // V(2\lambda_2 + \lambda_3).$$

Using the structure of $V(2\lambda_2 + \lambda_3)$ obtained from Jantzen's sum formula, and the observation that

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_2 + \lambda_4), L(\lambda_2)) \cong 0,$$

(by Schur's lemma) we obtain

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_2+\lambda_3)\neq 0.$$

However, the corresponding good filtration then gives

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_2+\lambda_3))$$

$$\precsim \mathrm{Hom}_G(V(\lambda_1+\lambda_2+\lambda_4), H^0(\lambda_2)\otimes H^0(\lambda_2+\lambda_3))\cong k.$$

x) $\mathrm{Soc}_{G_1}(L(\lambda_2)\otimes L(\lambda_1+\lambda_2+\lambda_4))\cong L(\lambda_1+\lambda_3)\oplus L(\lambda_2+\lambda_3+\lambda_4)$

The simple restricted module types that appear as G- composition factors are $L(\lambda_4)$, $L(\lambda_1+\lambda_3)$ and $L(\lambda_2+\lambda_3+\lambda_4)$ We have

$$\mathrm{Hom}_{G_1}(L(\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_2+\lambda_4))$$

$$\cong \mathrm{Hom}_{G_1}(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_1))\cong 0,$$

by Table 2-2 We also have

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_2+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_4))\cong 0,$$

by 3 2(r) Finally, we see that

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_2+\lambda_4)),\cong k,$$

by considering a good filtration and decomposition into linkage classes of

$$H^0(\lambda_2)\otimes H^0(\lambda_1+\lambda_2+\lambda_4) \cong L(\lambda_2)\otimes L(\lambda_2)//L(\lambda_2)\otimes L(2\lambda_1)//L(\lambda_2)\otimes L(\lambda_1+\lambda_2+\lambda_4)$$

$$\cong \{H^0(\lambda_1+2\lambda_2+\lambda_4)//H^0(2\lambda_1+\lambda_3+\lambda_4)//H^0(\lambda_2+\lambda_3+\lambda_4)\}$$
$$\oplus\{H^0(2\lambda_1+\lambda_2)//H^0(\lambda_1+2\lambda_4)//H^0(2\lambda_2)//H^0(\lambda_1+\lambda_3)\},$$

and by observing that

$$\mathrm{Hom}_G((L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_2)) \cong 0,$$

by Table 2-2

y) $\mathrm{Soc}_{G_1}(L(\lambda_2)\otimes L(\lambda_1+\lambda_3+\lambda_4)) \cong L(\lambda_2+\lambda_3)\oplus L(\rho)$.

The simple restricted module types that appear as G- composition factors are k, $L(\lambda_2+\lambda_3)$ and $L(\rho)$. We have

$$\mathrm{Hom}_{G_1}(k, L(\lambda_2)\otimes L(\lambda_1+\lambda_3+\lambda_4)) \cong 0,$$

by Schur's lemma. We also have

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_3), L(\lambda_2)\otimes L(\lambda_1+\lambda_3+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_2+\lambda_3)) \cong k,$$

by 3 2(w) above. Finally,

$$\mathrm{Hom}_{G_1}(L(\rho), L(\lambda_2) \otimes L(\lambda_1 + \lambda_3 + \lambda_4)) \cong k,$$

by Table 2-2, and the fact that $L(\rho)$ is injective for G_1

z) $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_1 + \lambda_2 + \lambda_3)) \cong L(\lambda_1 + \lambda_2) \oplus L(\lambda_1 + \lambda_3 + \lambda_4)$

The simple restricted module types that appear as G- composition factors are $L(\lambda_3)$, $L(\lambda_1 + \lambda_2)$ and $L(\lambda_1 + \lambda_3 + \lambda_4)$. We have

$$\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1 + \lambda_3 + \lambda_3))$$

$$\cong \mathrm{Hom}_G(L(\lambda_2 + \lambda_3 + \lambda_4), L(\lambda_2) \otimes L(\lambda_2)) \cong 0,$$

by Table 2-2, and

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_2), L(\lambda_2) \otimes L(\lambda_1 + \lambda_2 + \lambda_3)$$

$$\cong \mathrm{Hom}_G(L(\lambda_3 + \lambda_3 + \lambda_4), L(\lambda_2) \otimes L(\lambda_3 + \lambda_4)) \cong k,$$

by 3 2(u) above, and

$$\mathrm{Hom}_G(L(\lambda_1 + \lambda_3 + \lambda_4), L(\lambda_2) \otimes L(\lambda_1 + \lambda_2 + \lambda_3)$$

$$\cong \mathrm{Hom}_G(L(\lambda_2 + \lambda_3 + \lambda_4), L(\lambda_2) \otimes (L(\lambda_1 + \lambda_3 + \lambda_4)) \cong k,$$

by 3 2(x) above

aa) $\mathrm{Soc}_{G_1}(L(\lambda_2) \otimes L(\lambda_2 + \lambda_3 + \lambda_4)) \cong L(\lambda_2 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_3)$

The simple restricted module types that appear as G- composition factors are $L(\lambda_1)$, $L(\lambda_2 + \lambda_4)$ and $L(\lambda_1 + \lambda_2 + \lambda_3)$. We have

$$\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_2 + \lambda_3 + \lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_3), L(\lambda_2)\otimes L(\lambda_4)) \cong 0,$$

by Table 2-2, and

$$\mathrm{Hom}_G(L(\lambda_2+\lambda_4), L(\lambda_2)\otimes L(\lambda_2+\lambda_3+\lambda_4))$$

$$\cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_3), L(\lambda_2)\otimes L(\lambda_1+\lambda_3)) \cong k,$$

by 3 2(t) above.

Finally, we show

$$\mathrm{Hom}_G(L(\lambda_1+\lambda_2+\lambda_3), L(\lambda_2)\otimes L(\lambda_2+\lambda_3+\lambda_4)), \cong k,$$

considering a good filtration and decomposition into linkage classes of

$$H^0(\lambda_2)\otimes H^0(\lambda_2+\lambda_3+\lambda_4) \cong L(\lambda_2)\otimes L(\lambda_2+\lambda_3+\lambda_4)$$

$$\cong \{H^0(2\lambda_2+\lambda_3+\lambda_4)//H^0(\lambda_1+2\lambda_3+\lambda_4)//H^0(\lambda_1+\lambda_2+2\lambda_4)//H^0(\lambda_1+\lambda_2+\lambda_3)\}$$
$$\oplus\{H^0(\lambda_3+2\lambda_4)//H^0(2\lambda_3)//H^0(\lambda_2+\lambda_4)\}.$$

bb) $\mathrm{Soc}_{G_1}(L(\lambda_2)\otimes L(\rho)) \cong L(\lambda_1+\lambda_2+\lambda_4)$.

We have

$$\mathrm{Hom}_G(L(\nu^0), L(\lambda_2)\otimes L(\rho)) \cong \mathrm{Hom}_G(L(\rho), L(\lambda_2)\otimes L(\nu^0)^*) \cong 0,$$

for all restricted weights ν^0 except possibly $\nu^0 = \lambda_1+\lambda_2+\lambda_4$ by Table 2-2, but

$$\mathrm{Hom}_G(L(\rho), L(\lambda_2) \otimes L(\lambda_1 + \lambda_3 + \lambda_4)^*) \cong k$$

by 3.2(γ) above.

§3.3 Socles of Tensor Products for D_4

We now use the lemmas of section 3.1 to compute the D_1 socles of the tensor products of either $L(\delta_1)$ or $L(\delta_2)$ with the other restricted simple modules. Using the fact that there are 15 non-trivial restricted simple modules, and exploiting symmetry of D_4, (e.g. we do not have to compute both $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_3))$ and $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_4))$), we now list $(15-4)+6 = 17$ such computations.

a) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_1)) \cong k$.

By Table 2-1, the only simple restricted D-module types that appear as D- composition factors are $k, L(\delta_2)$. We have

$$\mathrm{Hom}_D(k, L(\delta_1) \otimes L(\delta_1)) \cong \mathrm{Hom}_D(L(\delta_1), L(\delta_1)) \cong k,$$

by Schur's lemma, and

$$\mathrm{Hom}_D(L(\delta_2), L(\delta_1) \otimes L(\delta_1)) \cong \mathrm{Hom}_D(L(\delta_1), L(\delta_2) \otimes L(\delta_1)) \cong 0,$$

by Table 2-1.

b) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_3)) \cong L(\delta_4)$.

By Table 2-1, the simple restricted module types that appear as D- composition factors are $L(\delta_1 + \delta_3), L(\delta_4)$. We have

$$\mathrm{Hom}_D(L(\delta_1 + \delta_3), L(\delta_1) \otimes L(\delta_3)) \cong \mathrm{Hom}_D(L(\delta_1), L(\delta_1 + \delta_3) \otimes L(\delta_3)) \cong 0,$$

by Table 2-1, and

$$\mathrm{Hom}_D(L(\delta_4), L(\delta_1) \otimes L(\delta_3)) \cong k,$$

by considering the good filtration of

$$L(\delta_1) \otimes L(\delta_3) \cong H^0(\delta_1) \otimes H^0(\delta_3) \cong H^0(\delta_1 + \delta_3)//H^0(\delta_4).$$

c) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_2)) \cong L(\delta_3 + \delta_4) \oplus L(\delta_1 + \delta_2)$

This is immediate from Table 2-1, since $\delta_3 + \delta_4$ and $\delta_1 + \delta_2$ lie in distinct linkage classes

d) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_3 + \delta_4)) \cong L(\delta_2)$

The simple restricted module types that appear as D- composition factors are k, $L(\delta_2), L(\delta_1 + \delta_3 + \delta_4)$ We have

$$\mathrm{Hom}_D(k, L(\delta_1) \otimes L(\delta_3 + \delta_4)) \cong \mathrm{Hom}_D(L(\delta_1), L(\delta_3 + \delta_4)) \cong 0,$$

by Schur's Lemma,

$$\mathrm{Hom}_D(L(\delta_1 + \delta_3 + \delta_4), L(\delta_1) \otimes L(\delta_3 + \delta_4))$$

$$\cong \mathrm{Hom}_D(L(\delta_3 + \delta_4), L(\delta_1 + \delta_3 + \delta_4) \otimes L(\delta_1)) \cong 0,$$

by Table 2-1, and finally,

$$\mathrm{Hom}_D(L(\delta_2), L(\delta_1) \otimes L(\delta_3 + \delta_4)) \cong \mathrm{Hom}_D(L(\delta_3 + \delta_4), L(\delta_1) \otimes L(\delta_2)) \cong k,$$

by 3.3(c) above.

e) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_1 + \delta_3)) \cong L(\delta_2 + \delta_4)$.

By Table 2-1, the only simple restricted module type that appears as a D- composition factor is $L(\delta_2 + \delta_4)$. We have

$$\mathrm{Hom}_D(L(\delta_2 + \delta_4), L(\delta_1) \otimes L(\delta_1 + \delta_3)) \cong \mathrm{Hom}_D(L(\delta_1 + \delta_3), L(\delta_1) \otimes L(\delta_2 + \delta_4)) \cong k,$$

by considering the good filtration of

$L(\delta_1) \otimes L(\delta_2 + \delta_4) \cong H^0(\delta_1) \otimes H^0(\delta_2 + \delta_4) \cong H^0(\delta_1 + \delta_2 + \delta_4) // H^0(\delta_2 + 2\delta_4) // H^0(\delta_2 + \delta_3) // H^0(\delta_1 + \delta_4)$

f) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_1 + \delta_2)) \cong L(\delta_2)$.

The simple restricted module types that appear as D- composition factors are k, $L(\delta_2)$, $L(\delta_1 + \delta_3 + \delta_4)$. We have

$$\mathrm{Hom}_D(k, L(\delta_1) \otimes L(\delta_1 + \delta_2)) \cong \mathrm{Hom}_D(L(\delta_1), L(\delta_1 + \delta_2)) \cong 0,$$

by Schur's Lemma,

$$\mathrm{Hom}_D(L(\delta_1 + \delta_3 + \delta_4), L(\delta_1) \otimes L(\delta_1 + \delta_2))$$

$$\cong \mathrm{Hom}_D(L(\delta_1 + \delta_2), L(\delta_1) \otimes L(\delta_1 + \delta_3 + \delta_4)) \cong 0,$$

by Table 2-1, and finally,

$$\mathrm{Hom}_D(L(\delta_2), L(\delta_1) \otimes L(\delta_1 + \delta_2)) \cong \mathrm{Hom}_D(L(\delta_1 + \delta_2), L(\delta_1) \otimes L(\delta_2)) \cong k,$$

by 3.3(c) above.

g) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_2 + \delta_3)) \cong L(\delta_1 + \delta_3) \oplus L(\delta_2 + \delta_4)$.

By Table 2-1, the simple restricted module types that appear as D composition factors are $L(\delta_4), L(\delta_1 + \delta_2), L(\delta_2 + \delta_4)$, and $L(\delta_1 + \delta_2 + \delta_3)$. We have

$$\mathrm{Hom}_D(L(\delta_4), L(\delta_1) \otimes L(\delta_2 + \delta_3)) \cong \mathrm{Hom}_D(L(\delta_2 + \delta_3), L(\delta_1) \otimes L(\delta_4)) \cong 0,$$

by 3.3(b) above,

$$\mathrm{Hom}_D(L(\delta_1 + \delta_2 + \delta_3), L(\delta_1) \otimes L(\delta_2 + \delta_3))$$

$$\cong \mathrm{Hom}_D(L(\delta_2 + \delta_3), L(\delta_1) \otimes L(\delta_1 + \delta_2 + \delta_3)) \cong 0,$$

by Table 2-1,

$$\mathrm{Hom}_D(L(\delta_1 + \delta_3), L(\delta_1) \otimes L(\delta_2 + \delta_3))$$

$$\cong \mathrm{Hom}_D(L(\delta_2 + \delta_3), L(\delta_1) \otimes L(\delta_1 + \delta_3)) \cong k$$

by 3.3(e) above, and

$$\mathrm{Hom}_D(L(\delta_2 + \delta_4), L(\delta_1) \otimes L(\delta_2 + \delta_3)) \cong k,$$

by considering a good filtration and decomposition into linkage classes of $L(\delta_1) \otimes L(\delta_2 + \delta_3) \cong H^0(\delta_1) \otimes H^0(\delta_2 + \delta_3) \cong \{H^0(\delta_1 + \delta_2 + \delta_3) // H^0(\delta_1 + \delta_3)\} \oplus \{H^0(\delta_4 + 2\delta_3) // H^0(\delta_2 + \delta_4)\}$.

h) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_1 + \delta_3 + \delta_4)) \cong L(\delta_2 + \delta_3 + \delta_4)$.

The simple restricted module types that appear as D- composition factors are $L(\delta_1)$ and $L(\delta_2 + \delta_3 + \delta_4)$. We have

$$\mathrm{Hom}_D(L(\delta_1), L(\delta_1) \otimes L(\delta_1 + \delta_3 + \delta_4)) \cong \mathrm{Hom}_D(L(\delta_1 + \delta_3 + \delta_4), L(\delta_1) \otimes L(\delta_1)) \cong 0,$$

by Table 2-1, and

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_1)\otimes L(\delta_1+\delta_3+\delta_4)) \cong k,$$

since it must be nonzero by the above and there are only 2 composition factors of $L(\delta_1)\otimes L(\delta_1+\delta_3+\delta_4))$ isomorphic to $L(\delta_2+\delta_3+\delta_4)$ (so that, if both of these factors are in the socle, we would have a D -composition series with either $L(\delta_1)$ or $L(\delta_3+\delta_4)$ appearing in the head, but $L(\delta_1)\otimes L(\delta_1+\delta_3+\delta_4))$ is self-dual.)

i) $\mathrm{Soc}_{D_1}(L(\delta_1)\otimes L(\delta_2+\delta_3+\delta_4)) \cong L(\delta_1+\delta_3+\delta_4)\oplus L(\rho)$

The simple restricted D-module types that appear as D_1- composition factors are $k, L(\delta_3), L(\delta_1+\delta_3+\delta_4)$, and $L(\rho)$. We have

$$\mathrm{Hom}_D(k, L(\delta_1)\otimes L(\delta_2+\delta_3+\delta_4)) \cong \mathrm{Hom}_D(L(\delta_1), L(\delta_2+\delta_3+\delta_4)) \cong 0,$$

by Schur's Lemma,

$$\mathrm{Hom}_D(L(\delta_3), L(\delta_1)\otimes L(\delta_2+\delta_3+\delta_4))$$

$$\cong \mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_1)\otimes L(\delta_3)) \cong 0,$$

by Table 2-1,

$$\mathrm{Hom}_D(L(\delta_1+\delta_3+\delta_4), L(\delta_1)\otimes L(\delta_2+\delta_3+\delta_4))$$

$$\cong \mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_1)\otimes L(\delta_1+\delta_3+\delta_4)) \cong k,$$

by 3.3(h) above, and finally,

$$\mathrm{Hom}_{D_1}(L(\rho), L(\delta_1) \otimes L(\delta_2 + \delta_3 + \delta_4)) \cong k,$$

by Table 2-1, and the fact that $L(\rho)$ is injective (and projective) for D_1

j) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\delta_1 + \delta_2 + \delta_4)) \cong L(\delta_1 + \delta_2 + \delta_3)$.

The simple restricted module types that appear as D- composition factors are $L(\delta_4)$, $L(\delta_1 + \delta_3)$ and $L(\delta_1 + \delta_2 + \delta_3)$. We have

$$\mathrm{Hom}_D(L(\delta_4), L(\delta_1) \otimes L(\delta_1 + \delta_2 + \delta_4)) \cong \mathrm{Hom}_D(L(\delta_1 + \delta_2 + \delta_4), L(\delta_1) \otimes L(\delta_4)) \cong 0$$

by Table 2-1,

$$\mathrm{Hom}_D(L(\delta_1 + \delta_3), L(\delta_1) \otimes L(\delta_1 + \delta_2 + \delta_4))$$

$$\cong \mathrm{Hom}_D(L(\delta_1 + \delta_2 + \delta_4), L(\delta_1) \otimes L(\delta_1 + \delta_3)) \cong 0,$$

by Table 2 1, and so, we must have

$$0 \neq \mathrm{Hom}_D(L(\delta_1 + \delta_2 + \delta_3), L(\delta_1) \otimes L(\delta_1 + \delta_2 + \delta_4)),$$

$$\precsim \mathrm{Hom}_D(V(\delta_1 + \delta_2 + \delta_3), H^0(\delta_1) \otimes H^0(\delta_1 + \delta_2 + \delta_4)) \cong k,$$

by considering a good filtration of

$H^0(\delta_1) \otimes H^0(\delta_1 + \delta_2 + \delta_4)) \cong H^0(2\delta_1 + \delta_2 + \delta_4)//H^0(2\delta_2 + \delta_4)//H^0(\delta_1 + \delta_3 + 2\delta_4)//H^0(\delta_1 + \delta_2 + \delta_3)//H^0(2\delta_1 + \delta_4)//H^0(\delta_2 + \delta_4)$.

k) $\mathrm{Soc}_{D_1}(L(\delta_1) \otimes L(\rho)) \cong L(\delta_2 + \delta_3 + \delta_4)$.

We have

$$\mathrm{Hom}_D(L(\nu^0), L(\delta_1) \otimes L(\rho)) \cong \mathrm{Hom}_D(L(\rho), L(\delta_1) \otimes L(\nu^0)) \cong 0,$$

for all restricted weights ν^0 except $\nu^0 = \delta_2 + \delta_2 + \delta_4$ by Table 2-1, and

$$\mathrm{Hom}_D(L(\rho), L(\delta_1) \otimes L(\delta_2 + \delta_3 + \delta_4)) \cong k$$

by 3.3(i) above.

l) $\mathrm{Soc}_{D_1}(L(\delta_2) \otimes L(\delta_2)) \cong k \oplus L(\delta_2)$.

This was computed by Sin [20]

m) $\mathrm{Soc}_{D_1}(L(\delta_2) \otimes L(\delta_3 + \delta_4)) \cong L(\delta_1) \oplus L(\delta_1 + \delta_2)$.

The simple restricted module types that appear as D- composition factors are $L(\delta_1)$, $L(\delta_1 + \delta_2)$, and $L(\delta_2 + \delta_3 + \delta_4)$. We have

$$\mathrm{Hom}_D(L(\delta_1), L(\delta_2) \otimes L(\delta_3 + \delta_4)) \cong \mathrm{Hom}_D(L(\delta_3 + \delta_4), L(\delta_1) \otimes L(\delta_2)) \cong k,$$

by 3.3(c) above. Also,

$$\mathrm{Hom}_D(L(\delta_1 + \delta_2), L(\delta_2) \otimes L(\delta_2 + \delta_4)) \cong k.$$

This follows by considering the decomposition of $L(\delta_2) \otimes L(\delta_3 + \delta_4)$ into linkage classes The only other composition factors with highest weight in the same linkage class as $\delta_1 + \delta_2$ are $L(\delta_1 + 2\delta_3)$ and $L(\delta_1 + 2\delta_4)$, but these do not appear in the D-socle. (For example,

$$\mathrm{Hom}_D(L(\delta_1 + 2\delta_3), L(\delta_2) \otimes L(\delta_3 + \delta_4))$$

$$\cong \mathrm{Hom}_G(L(\delta_3 + \delta_4 + 2\delta_3), L(\delta_2) \otimes L(\delta_1)) \cong 0,$$

by Table 2-1) Therefore, we must have

$$\mathrm{Hom}_D(L(\delta_1 + \delta_3), L(\delta_2) \otimes L(\delta_3 + \delta_4)) \neq 0,$$

but there are only 2 composition factors of $L(\delta_2) \otimes L(\delta_3 + \delta_4)$ that are isomorphic to $L(\delta_1 + \delta_2)$ so that if both appear in the D socle, we would have a composition series of $L(\delta_2) \otimes L(\delta_3 + \delta_4)$ with either $L(\delta_1 + 2\delta_2)$ or $L(\delta_1 + 2\delta_4)$ appearing in the head. Finally,

$$\mathrm{Hom}_D(L(\delta_2 + \delta_3 + \delta_4), L(\delta_2) \otimes L(\delta_3 + \delta_4)) \cong 0,$$

again by considering the decomposition into linkage classes. If the unique composition factor isomorphic to $L(\delta_2 + \delta_3 + \delta_4)$ appeared in the socle, we would have a composition series in which 2 copies of $L(\delta_1)$ appeared in the head (since $\mathrm{Ext}^1_D(L(\delta_1), L(\delta_1)) \cong 0$) contradicting

$$\mathrm{Hom}_D(L(\delta_2) \otimes L(\delta_3 + \delta_4), L(\delta_1))$$

$$\cong \mathrm{Hom}_D(L(\delta_1), L(\delta_2) \otimes L(\delta_3 + \delta_4)) \cong k.$$

(See above.)

n) $\mathrm{Soc}_{D_1}(L(\delta_2) \otimes L(\delta_1 + \delta_2)) \cong L(\delta_1) \oplus L(\delta_3 + \delta_4) \oplus L(\delta_1 + \delta_2) \oplus L(\delta_2 + \delta_3 + \delta_4)$. The simple restricted module types that appear as D composition factors are $L(\delta_1)$, $L(\delta_3 + \delta_4)$, $L(\delta_1 + \delta_2)$, and $L(\delta_2 + \delta_3 + \delta_4)$. We have

$$\mathrm{Hom}_D(L(\delta_1), L(\delta_2) \otimes L(\delta_1 + \delta_2)) \cong \mathrm{Hom}_D(L(\delta_1 + \delta_2), L(\delta_1) \otimes L(\delta_2)) \cong k,$$

by 3.3(c) above,

$$\mathrm{Hom}_D(L(\delta_3 + \delta_4), L(\delta_2) \otimes L(\delta_1 + \delta_2)) \cong \mathrm{Hom}_D(L(\delta_1 + \delta_2), L(\delta_2) \otimes L(\delta_3 + \delta_4)) \cong k,$$

by 3.3(m) above,

$$\mathrm{Hom}_D(L(\delta_1 + \delta_2), L(\delta_2) \otimes L(\delta_1 + \delta_2)) \cong k,$$

by an argument similar to that used in 3.3(m). Finally,

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_1+\delta_2))$$

$$\cong \mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)) \cong k,$$

again, by the argument of 3.3(m).

o) $\mathrm{Soc}_{D_1}(L(\delta_2)\otimes L(\delta_1+\delta_3+\delta_4)) \cong L(\delta_1+\delta_3+\delta_4)\oplus L(\rho)$.

This was computed by Siu [20].

p) $\mathrm{Soc}_{D_1}(L(\delta_2)\otimes L(\rho)) \cong L(\delta_1+\delta_3+\delta_4)\oplus 2L(\rho)$.

We have

$$\mathrm{Hom}_D(L(\nu^0), L(\delta_2)\otimes L(\rho)) \cong \mathrm{Hom}_D(L(\rho), L(\delta_2)\otimes L(\nu^0)) \cong 0,$$

for all restricted weights ν^0 except $\nu^0 = \delta_1+\delta_3+\delta_4$ and $\nu^0 = \rho$ by Table 2-1,

$$\mathrm{Hom}_D(L(\rho), L(\delta_2)\otimes L(\delta_1+\delta_3+\delta_4)) \cong k$$

by 3.3(o) above, and $L(\rho)$ is injective for D_1. (See table 2-1.)

q) $\mathrm{Soc}_{D_1}(L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)) \cong L(\delta_1+\delta_2)\oplus L(\delta_2+\delta_3+\delta_4)$.

The simple restricted module types that appear as D- composition factors are $L(\delta_1)$, $L(\delta_3+\delta_4)$, $L(\delta_1+\delta_2)$, and $L(\delta_2+\delta_3+\delta_4)$. We have

$$\mathrm{Hom}_D(L(\delta_1), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4))$$

$$\cong \mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_1)\otimes L(\delta_2)) \cong 0,$$

by Table 2-1,

$$\mathrm{Hom}_D(L(\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4))$$

$$\cong \mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_3+\delta_4)\cong 0,$$

by 3.3(m) above,

$$\mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4))\cong k,$$

by an argument similar to that used in 3.3(m). Finally, we compute

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)),$$

in the next subsection, along a couple of miscellaneous lemmas about D-socles of certain tensor products for later use.

§3.4 Miscellaneous Socle Lemmas

We include here some miscellaneous lemmas involving some socles of tensor products which will be used later. The next lemma will actually follow from the results of the next chapter, but is included here mainly for the sake of organization.

Lemma 3.4.1 *Let D (resp. G) be the simply connected algebraic group of type D_4 (resp. A_4). Let M be any composition factor of $E = \mathrm{Ext}^1_{D_1}(L(\mu^0), L(\nu^0))^{(2^{-1})}$ where μ^0, ν^0 are any two restricted weights. If (λ, β) is any pair of 2 restricted weights, then*

$$\mathrm{Hom}_{D_1}(L(\lambda), M\otimes L(\beta))$$

is D-trivial. The analogous result holds for $G = A_4$.

Proof. In chapter 4 we will show that the only composition factors that occur in $E = \mathrm{Ext}^1_{D_1}(L(\mu^0), L(\nu^0))^{(2^{-1})}$ are those for which the hypothesis of Lemma 3.1.2 applies. ∎

In the following, denote by G the simply connected group of type B_4, by $\tilde{G}$ the simply connected group of type C_4, and by D the subgroup of G generated by the long root subgroups (which is simply connected of type D_4)

Lemma 3.4.2.

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)) \cong k$$

Proof First we observe that

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)) \neq 0$$

This follows from the fact that

$$\mathrm{Hom}_D(L, L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4))$$

can be shown to be zero for all other composition factors of $L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)$ in the same linkage class as $L(\delta_2+\delta_3+\delta_4)$

We have

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4))$$

$$\cong \mathrm{Hom}_{D_1}(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)),$$

(by dimension argument)

$$\cong \mathrm{Hom}_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3), \tilde{L}(\tilde{\omega}_2)\otimes \tilde{L}(\omega_2+\tilde{\omega}_3))$$

$$\cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3), \tilde{L}(\tilde{\omega}_2)\otimes \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)),$$

(by dimension argument)

$$\cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\tilde{\omega}_2), \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3) \otimes \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))$$

$$\preceq \mathrm{Hom}_{\tilde{G}}(\tilde{V}(\tilde{\omega}_3), \tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3) \otimes \tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3))$$

$$\cong \mathrm{Hom}_{\tilde{G}}(\tilde{V}(\tilde{\omega}_2+\tilde{\omega}_3), \tilde{H}^0(\tilde{\omega}_2) \otimes \tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)) \cong 2k,$$

by considering a good filtration of

$$\tilde{H}^0(\tilde{\omega}_2) \otimes \tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_2)$$

$\cong \tilde{H}^0(2\tilde{\omega}_2+\tilde{\omega}_2)//\tilde{H}^0(\tilde{\omega}_1+2\tilde{\omega}_3)//\tilde{H}^0(\tilde{\omega}_1+\tilde{\omega}_2+\tilde{\omega}_4)//\tilde{H}^0(\tilde{\omega}_3+\tilde{\omega}_4)//\tilde{H}^0(\tilde{\omega}_1+2\tilde{\omega}_2)//$
$\tilde{H}^0(2\tilde{\omega}_1+\tilde{\omega}_3)//2\tilde{H}^0(\omega_2+\tilde{\omega}_3)//H^0(\tilde{\omega}_1+\tilde{\omega}_4)//\tilde{H}^0(\tilde{\omega}_1+\tilde{\omega}_2)//\tilde{H}^0(\tilde{\omega}_3)$.

On the other hand, there is an embedding of $\tilde{V}(\tilde{\omega}_3)$ into $\tilde{V}(2\tilde{\omega}_1)$, which in turn can be embedded into $\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)\otimes\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$ (which embeds into $\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)\otimes\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)$) This is because the morphism of k-group schemes

$$\tilde{G} \hookrightarrow GL_k(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))$$

is injective (since the weights of $\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$ generate the character group of $\tilde{T}$) and therefore induces an embedding of $\tilde{G}$ modules

$$\tilde{\mathfrak{g}} \hookrightarrow End_k(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))$$

(where $End_k(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)) \cong \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)^* \otimes \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$, and $\tilde{\mathfrak{g}} \cong \tilde{V}(2\tilde{\omega}_1)$.) Therefore, we must have

$$\dim_k(\mathrm{Hom}_{\tilde{G}}(\tilde{L}(\tilde{\omega}_2), \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3) \otimes \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))) \leq$$

$$\dim_k(\mathrm{Hom}_{\tilde{G}}(\tilde{L}(\tilde{\omega}_2), \tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3) \otimes \tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3))) \leq 1.$$

Lemma 3.4.3.

$$Hom_D(L(\delta_2+\delta_4), L(\delta_1+\delta_4) \otimes L(\delta_1+\delta_2)) \neq 0$$

Proof. This follows from the fact that

$$\mathrm{Hom}_D(L, L(\delta_1+\delta_4) \otimes L(\delta_1+\delta_2))$$

can be shown to be zero for all other composition factors of $L(\delta_1+\delta_4) \otimes L(\delta_1+\delta_2)$ in the same linkage class as $L(\delta_2+\delta_4)$.

Lemma 3.4.4.

$$Hom_D(L(\delta_1+\delta_2+\delta_4), L(\delta_1+\delta_2+\delta_4) \otimes L(\delta_2+\delta_3)) \neq 0$$

Proof. This follows from the fact that

$$\mathrm{Hom}_D(L, L(\delta_1+\delta_3+\delta_4) \otimes L(\delta_2+\delta_3))$$

can be shown to be zero for all other composition factors of $L(\delta_1+\delta_2+\delta_4) \otimes L(\delta_2+\delta_3)$ in the same linkage class as $L(\delta_1+\delta_2+\delta_4)$. This follows immediately from Table 2-1 (and the identity $\mathrm{Hom}_D(U, V \otimes W) \cong \mathrm{Hom}_D(W^*, V \otimes U^*)$), except for the composition factors $L(\delta_1+\delta_2+2\delta_3+\delta_4)$ (Remark: $L(\delta_2+3\delta_3)$ is in a different linkage class.) However, we obtain

$$\mathrm{Hom}_D(L(\delta_1+\delta_2+2\delta_3+\delta_4), L(\delta_1+\delta_3+\delta_4) \otimes L(\delta_2+\delta_3)) \cong 0,$$

from the inclusion

$$V(\delta_1 + \delta_2 + 2\delta_3 + \delta_4) \subseteq V(\delta_1 + \delta_3 + \delta_4) \otimes V(\delta_2 + \delta_3),$$

which has a filtration

$$L(\delta_1 + \delta_3 + \delta_4) \otimes L(\delta_2 + \delta_3) // \text{rad}[V(\delta_1 + \delta_2 + \delta_4)] \otimes L(\delta_2 + \delta_3)$$

We then observe from Table 2-1, and the composition factors of $\text{rad}[V(\delta_1 + \delta_3 + \delta_4)]$ that $\text{rad}[V(\delta_1 + \delta_3 + \delta_4)] \otimes L(\delta_2 + \delta_3)$ contains no composition factor isomorphic to $L(\delta_1 + 2\delta_2 + \delta_4)$, which is a composition factor of $\text{rad}[V(\delta_1 + \delta_2 + 2\delta_3 + \delta_4)]$. Therefore, the quotient of $V(\delta_1 + \delta_2 + 2\delta_3 + \delta_4)$ inside $L(\delta_1 + \delta_3 + \delta_4) \otimes L(\delta_2 + \delta_3)$ has nonzero radical and so the unique composition factor of $L(\delta_1 + \delta_3 + \delta_4) \otimes L(\delta_2 + \delta_3)$ isomorphic to $L(\delta_1 + \delta_2 + 2\delta_3 + \delta_4)$ cannot appear in the socle of $L(\delta_1 + \delta_3 + \delta_4) \otimes L(\delta_2 + \delta_3)$.

CHAPTER 4

MODULE EXTENSIONS FOR INFINITESIMAL SUBGROUPS

§4.1. Extensions for $(A_4)_1$

In this section, we compute, using the strategy described previously, all of the quantities $\mathrm{Ext}^1_{G_1}(L, M)$, where L and M are restricted simple modules. Utilizing Lemma 3.1.1 together with Table 2-2 and exploiting the symmetry of A_4, we need to list 42 such computations. We also use the fact that for $G = A_4$, $\mathrm{Ext}^1_{G_1}(L, L) = 0$ for any restricted simple module L.

a) $\mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \cong 0$.

By Lemma 3.1.1 and Table 2-2, the only simple G-module that could appear as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3))$ is $L(2\lambda_3)$.

i) $\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)))$

$$\cong \mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_4), L(\lambda_3) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + \lambda_4 + 2\lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_1)) \cong 0,$$

since $\lambda_1 + 3\lambda_4$ is not comparable with $3\lambda_1 + \lambda_2 + \lambda_3$ in the (usual) partial order.

b) $\mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$.

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)))$

$$\cong \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2)\otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+\lambda_4+2\lambda_3), L(\lambda_1+\lambda_2+\lambda_3+2\lambda_1)) \cong 0,$$

since $3\lambda_3+\lambda_4$ is not comparable with $3\lambda_1+\lambda_2+\lambda_3$ in the (usual) partial order

c) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_1+\lambda_2+\lambda_3)) \cong L(2\lambda_4)$

The only simple G-module that could appear as a composition factor is $L(2\lambda_4)$.

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_1+\lambda_2+\lambda_3)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_2+2\lambda_4), L(\lambda_1+\lambda_2+\lambda_3)) \cong k,$$

by considering the Weyl module $V(\lambda_1+\lambda_2+2\lambda_4)$

d) $\mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_3)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_1)$.

i) $\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_3)))$

$$\cong \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_3))\otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1)\otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+\lambda_3+\lambda_4+2\lambda_2), L(\lambda_1+\lambda_2+\lambda_3+2\lambda_1)) \cong 0,$$

since $3\lambda_2+\lambda_3+\lambda_4$ is not comparable with $3\lambda_1+\lambda_2+\lambda_3$ in the (usual) partial order

e) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_1+\lambda_2+\lambda_4)) \cong L(2\lambda_1)$.

From the fact observed by Andersen [2, 4] and Jantzen [4, 13] that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_1+\lambda_2+\lambda_4))^{(2^{-1})} \cong 0$. Then from the short exact sequence

$$0 \to L(\lambda_1+\lambda_2+\lambda_4) \to H^0(\lambda_1+\lambda_2+\lambda_4) \to \begin{matrix} L(\lambda_2) \\ L(2\lambda_1) \end{matrix} \to 0,$$

we obtain

$$0 \to L(2\lambda_1) \to H^1(G_1, L(\lambda_1+\lambda_2+\lambda_4)) \to 0,$$

by taking G_1-fixed points, and considering the long exact sequence in cohomology.

f) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_3)$.

i) $\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \cong k,$$

by considering the Weyl module $V(\lambda_1+2\lambda_3)$.

g) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$.

The only simple G-module isomorphism types that could appear as a composition factors are $L(2\lambda_4), L(2[\lambda_1+\lambda_3])$.

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4+2\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0,$$

as $3\lambda_4$ is not comparable with $\lambda_1+\lambda_2+\lambda_4$ in the (usual) partial order.

ii) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2[\lambda_1+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_3) \otimes L(\lambda_4)) \cong k$.)

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2[\lambda_1 + \lambda_2]), L(\lambda_1 + \lambda_2 + \lambda_4 + 2\lambda_4)) \cong 0,$$

since $2\lambda_1 + 2\lambda_2 + \lambda_4$ is not comparable with $\lambda_1 + \lambda_2 + 3\lambda_4$ in the (usual) partial order.

h) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong 0$

The only simple G-module isomorphism types that could appear as a composition factors are k, $L(2[\lambda_1 + \lambda_4])$.

i) $\mathrm{Hom}_G(k, \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_2 + \lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_2), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong 0,$$

by considering the Weyl module $V(\lambda_1 + \lambda_2 + \lambda_4) \subseteq V(\lambda_1 + \lambda_2) \otimes V(\lambda_4) \cong L(\lambda_1 + \lambda_2) \otimes L(\lambda_4)$. Since $\mathrm{Hom}_G(L(2\lambda_1), L(\lambda_1 + \lambda_2) \otimes L(\lambda_4)) \cong \mathrm{Hom}_G(L(2\lambda_1 + \lambda_3 + \lambda_4), L(\lambda_4)) \cong 0$, we must have that the single composition factor of $V(\lambda_1 + \lambda_2 + \lambda_4)$ isomorphic to $L(\lambda_2)$ lies in $\mathrm{Rad}^2(V(\lambda_1 + \lambda_2 + \lambda_4))$.

ii) $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_2 + \lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_2 + \lambda_4)) \otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_4) \otimes L(\lambda_2)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2[\lambda_3 + \lambda_4]), L(\lambda_1 + \lambda_2 + \lambda_4 + 2\lambda_2)) \cong 0,$$

since $\lambda_2 + 2\lambda_3 + 2\lambda_4$ is not comparable with $\lambda_1 + 3\lambda_2 + \lambda_4$ in the (usual) partial order

i) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong 0$

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_1 + \lambda_2 + \lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_1 + \lambda_2 + \lambda_4)) \otimes L(2[\lambda_1 + \lambda_4]))$$

(as $\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+2[\lambda_3+\lambda_4]), L(\lambda_1+\lambda_2+\lambda_4+2[\lambda_1+\lambda_4])) \cong 0,$$

since $3\lambda_3+2\lambda_4$ is not comparable with $3\lambda_1+\lambda_2+2\lambda_4$ in the (usual) partial order

j) $\mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$

The only simple G-module that could appear as a composition factor is $L(2\lambda_3)$

i) $\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_3)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+\lambda_4+2\lambda_2), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $3\lambda_2+\lambda_4$ is not comparable with $\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order.

k) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$

The only simple G-module that could appear as a composition factor is $L(2\lambda_4)$

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_3+2\lambda_3), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+3\lambda_3$ is not comparable with $\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order

l) $\mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$

The only simple G-module that could appear as a composition factor is $L(2\lambda_1)$

i) $\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+\lambda_3+2\lambda_2), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_1)) \cong 0,$$

since $3\lambda_2+\lambda_3$ is not comparable with $3\lambda_1+\lambda_2+\lambda_4$ in the (usual) partial order

m) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0.$

The only simple G module that could appear as a composition factor is $L(2\lambda_2)$.

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_3+\lambda_4+2\lambda_1), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $3\lambda_1+\lambda_3+\lambda_4$ is not comparable with $\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order

n) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_2+\lambda_3)) \cong F$

From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1, \lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_2+\lambda_3))^{(2^{-1})} \cong 0$. Then from the short exact sequence

$$0 \to L(\lambda_2+\lambda_3) \to H^0(\lambda_2+\lambda_3) \to k \to 0,$$

we obtain

$$0 \to k \to H^1(G_1, L(\lambda_2+\lambda_3)) \to 0,$$

by taking G_1 fixed points, and considering the long exact sequence in cohomology.

o) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)) \cong 0$

The only simple G-module types that could appear as composition factors are $L(2\lambda_2)$, $L(2[\lambda_3+\lambda_4])$

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2\lambda_1), L(\lambda_2+\lambda_3+2\lambda_4)) \cong 0,$$

since $3\lambda_1$ is not comparable with $\lambda_2+\lambda_3+2\lambda_4$ in the (usual) partial order

ii) $\mathrm{Hom}_G(L(2[\lambda_3+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)) \otimes L(2\lambda_3))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4) \otimes L(\lambda_3)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2[\lambda_1+\lambda_4]), L(\lambda_2+\lambda_3+2\lambda_3)) \cong 0,$$

since $3\lambda_1+2\lambda_4$ is not comparable with $\lambda_2+3\lambda_3$ in the (usual) partial order.

p) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_3)) \cong 0$.

The only simple G-module types that could appear as composition factors are $L(2\lambda_4)$, $L(2[\lambda_1+\lambda_3])$

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_3)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+2\lambda_4), L(\lambda_2+\lambda_3)) \cong 0,$$

by examining the Weyl modules $V(\lambda_2+2\lambda_4)$ and $V(2\lambda_1)$. From $\mathrm{Ext}^1_G(k, L(\lambda_2+2\lambda_4)) \cong \mathrm{Ext}^1_G(L(2\lambda_1), L(\lambda_2)) \cong k$, we must have that the radical of $V(\lambda_2+2\lambda_4)$ is uniserial of the form

$$\begin{matrix} k \\ L(\lambda_2+\lambda_3) \\ k \end{matrix}$$

ii) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_3)))$

$$\lesssim \mathrm{Hom}_G(L(2[\lambda_1+\lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_3)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_1+\lambda_3) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+2[\lambda_1+\lambda_2]), L(\lambda_2+\lambda_3+2\lambda_4)) \cong 0,$$

since $2\lambda_1+3\lambda_2$ is not comparable with $\lambda_2+\lambda_3+2\lambda_4$ in the (usual) partial order

q) $\mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_2+\lambda_3)) \cong 0$

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$.

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_2+\lambda_3)))$

$$\lesssim \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_2+\lambda_3)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_4)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\lambda_2 + \lambda_4 + 2\lambda_1), L(\lambda_2 + \lambda_3 + 2\lambda_4)) \cong 0,$$

since $2\lambda_1 + \lambda_2 + \lambda_4$ is not comparable with $\lambda_2 + \lambda_3 + 2\lambda_4$ in the (usual) partial order

r) $\operatorname{Ext}^1_{G_1}(k, L(\lambda_2 + \lambda_4)) \cong L(2\lambda_3)$

From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong ind_B^G(H^1(B_1, \lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_2 + \lambda_4))^{(2^{-1})} \cong L(\lambda_3)$. Then from the short exact sequence

$$0 \to L(\lambda_2 + \lambda_4) \to H^0(\lambda_2 + \lambda_4) \to L(\lambda_1) \to 0,$$

we obtain

$$0 \to H^1(G_1, L(\lambda_2 + \lambda_4)) \to L(2\lambda_3) \to 0,$$

by taking G_1-fixed points, and considering the long exact sequence in cohomology

s) $\operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)) \cong k$

The only possible isomorphism types of summands of the socle are $k, L(2[\lambda_1 + \lambda_4])$, and $L(2[\lambda_2 + \lambda_3])$

i) $\operatorname{Hom}_G(k, \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)))$

$$\cong \operatorname{Ext}^1_G(L(\lambda_1), L(\lambda_2 + \lambda_4)) \cong k,$$

by considering the Weyl module $V(\lambda_2 + \lambda_4)$

ii) $\operatorname{Hom}_G(L(2[\lambda_1 + \lambda_4]), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)))$

$$\cong \operatorname{Hom}_G(L(2[\lambda_3 + \lambda_4]), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_1+\lambda_4)\otimes L(\lambda_2)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2[\lambda_3+\lambda_4]), L(\lambda_2+\lambda_4+2\lambda_2)) \cong 0,$$

since $\lambda_1+2\lambda_3+2\lambda_4$ is not comparable with $3\lambda_2+\lambda_4$ in the (usual) partial order.

iii) $\mathrm{Hom}_G(L(2[\lambda_2+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_3+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4))\otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_2+\lambda_3)\otimes L(\lambda_2)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2[\lambda_3+\lambda_4]), L(\lambda_2+\lambda_4+2\lambda_2)) \cong 0,$$

since $\lambda_1+2\lambda_3+2\lambda_4$ is not comparable with $3\lambda_2+\lambda_4$ in the (usual) partial order.

We have thus shown that

$$\mathrm{soc}_G(\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4))) \cong k)$$

Now, the only type of simple G-module which can appear as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4))$, and which extends k, is $L(2[\lambda_2+\lambda_3])$. However, the argument of part (iii) shows that $L(2[\lambda_2+\lambda_3])$ cannot be a summand of the second socle layer because $L(\lambda_3+\lambda_4)$ does not extend $k\otimes L(\lambda_2)\cong L(\lambda_2)$.

f) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2+\lambda_4)) \cong L(2\lambda_1)$

The only possible isomorphism types of summands of the socle are $L(2\lambda_1)$ and $L(2[\lambda_2+\lambda_4])$.

i) $\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2+\lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2\lambda_1), L(\lambda_2 + \lambda_4)) \cong k,$$

by considering the Weyl module $V(\lambda_4 + 2\lambda_1)$.

ii) $\mathrm{Hom}_G(L(2[\lambda_2 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2 + \lambda_4)))$

$$\lesssim \mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3 + \lambda_4), L(\lambda_2 + \lambda_4) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2[\lambda_3 + \lambda_4]), L(\lambda_2 + \lambda_4 + 2\lambda_1)) \cong 0,$$

since $2\lambda_3 + 3\lambda_4$ is not comparable with $2\lambda_1 + \lambda_2 + \lambda_4$ in the (usual) partial order.

We have thus shown that

$$\mathrm{soc}_G(\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2 + \lambda_4))) \cong L(2\lambda_1).$$

Now, the only type of simple G-module which can appear as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2 + \lambda_4))$, and which extends $L(2\lambda_1)$, is $L(2[\lambda_2 + \lambda_4])$. However, the argument of part (ii) shows that $L(2[\lambda_2 + \lambda_4])$ cannot be a summand of the second socle layer because $L(\lambda_3 + \lambda_4)$ does not extend any of the composition factors of $L(\lambda_1) \otimes L(\lambda_1)$

ii) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_4)) \cong 0$

The only simple G-module types that could appear as composition factors are $L(2\lambda_2)$, $L(2[\lambda_3 + \lambda_4])$

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_4)))$

$$\lesssim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k,$)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2\lambda_3), L(\lambda_2 + \lambda_4 + 2\lambda_1)) \cong 0,$$

since $\lambda_2 + 2\lambda_3$ is not comparable with $2\lambda_1 + \lambda_2 + \lambda_4$ in the (usual) partial order

iii) $\mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_3 + \lambda_4) \otimes L(\lambda_1)) \cong \mathrm{Hom}_G(L(\lambda_1 + \lambda_2), L(\lambda_2) \otimes L(\lambda_1)) \cong k,$)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2\lambda_3), L(\lambda_2 + \lambda_4 + 2\lambda_1)) \cong 0,$$

since $\lambda_2 + 2\lambda_3$ is not comparable with $2\lambda_1 + \lambda_2 + \lambda_4$ in the (usual) partial order

v) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2 + \lambda_4)) \cong 0$

The only simple G module types that could appear as composition factors are $L(2\lambda_4)$, $L(2[\lambda_1 + \lambda_3])$

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4) \otimes L(\lambda_4)) \cong k,$)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3 + 2\lambda_3), L(\lambda_2 + \lambda_4 + 2\lambda_4)) \cong 0,$$

since $3\lambda_3$ is not comparable with $\lambda_2 + 3\lambda_4$ in the (usual) partial order

ii) $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1 + \lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_1+\lambda_3)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+2[\lambda_1+\lambda_2]), L(\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $2\lambda_1+2\lambda_2+\lambda_3$ is not comparable with $\lambda_2+3\lambda_4$ in the (usual) partial order

w) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_2+\lambda_4)) \cong 0$.

The only simple G module that could appear as a composition factor is $L(2\lambda_3)$

i) $\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_2+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_3)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_4+2\lambda_2), L(\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+2\lambda_2+\lambda_4$ is not comparable with $\lambda_2+3\lambda_4$ in the (usual) partial order

x) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4)) \cong 0$

The only simple G module types that could appear as composition factors are $L(2\lambda_1)$, $L(2[\lambda_2+\lambda_4])$

i) $\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4))\otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_1)\otimes L(\lambda_2)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_3+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_2)) \cong 0,$$

since $\lambda_1+3\lambda_3$ is not comparable with $3\lambda_2+\lambda_4$ in the (usual) partial order

ii) $\mathrm{Hom}_G(L(2[\lambda_2 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_3), L(\lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_3), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2 + \lambda_4) \otimes L(\lambda_2)) \cong \mathrm{Hom}_G(L(\lambda_1 + \lambda_3), L(\lambda_2) \otimes L(\lambda_2)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + \lambda_3 + 2\lambda_3), L(\lambda_2 + \lambda_4 + 2\lambda_2)) \cong 0,$$

since $\lambda_1 + 3\lambda_3$ is not comparable with $3\lambda_2 + \lambda_4$ in the (usual) partial order

y) $\mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_2 + \lambda_4)) \cong 0$

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3 + \lambda_4 + 2\lambda_3), L(\lambda_2 + \lambda_4 + 2\lambda_1)) \cong 0,$$

since $3\lambda_3 + \lambda_4$ is not comparable with $2\lambda_1 + \lambda_2 + \lambda_4$ in the (usual) partial order

z) $\mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_2 + \lambda_4)) \cong 0$

The only simple G module that could appear as a composition factor is $L(2\lambda_4)$

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_3), L(\lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_2+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+\lambda_2+2\lambda_3$ is not comparable with $\lambda_2+3\lambda_4$ in the (usual) partial order.

aa) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_3+\lambda_4)) \cong 0$.

From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong ind_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_3+\lambda_4))^{(2^{-1})} \cong 0$ However, $H^0(\lambda_3+\lambda_4) \cong L(\lambda_3+\lambda_4)$

bb) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_3+\lambda_4)) \cong 0$

The only simple G module that could appear as a composition factor is $L(2\lambda_3)$

i) $\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_3+\lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_3+\lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_3)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2\lambda_2), L(\lambda_3+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+2\lambda_2$ is not comparable with $\lambda_3+3\lambda_4$ in the (usual) partial order

cc) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_3+\lambda_4)) \cong 0$

The only simple G-module types that could appear as composition factors are $L(2\lambda_4)$, $L(2[\lambda_1+\lambda_3])$

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_3+\lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4+2\lambda_4), L(\lambda_3+\lambda_4)) \cong 0,$$

by considering the Weyl module $V(3\lambda_4)$

ii) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_3+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1+\lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_3+\lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_1+\lambda_3)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_4+2[\lambda_1+\lambda_2]), L(\lambda_3+\lambda_4+2\lambda_4)) \cong 0,$$

since $2\lambda_1+2\lambda_2+\lambda_4$ is not comparable with $\lambda_3+3\lambda_4$ in the (usual) partial order

dd) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3+\lambda_4)) \cong 0$.

The only simple G-modules that could appear as a composition factors are k and $L(2[\lambda_1+\lambda_4])$.

i) $\mathrm{Hom}_G(k, \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3+\lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_2), L(\lambda_3+\lambda_4)) \cong 0,$$

by considering the Weyl module $V(\lambda_3+\lambda_4) \cong L(\lambda_3+\lambda_4)$

ii) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1+\lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3+\lambda_4)) \otimes L(2\lambda_3))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_1+\lambda_4)\otimes L(\lambda_3)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+2[\lambda_1+\lambda_2]), L(\lambda_3+\lambda_4+2\lambda_3)) \cong 0,$$

since $2\lambda_1+3\lambda_2$ is not comparable with $3\lambda_3+\lambda_4$ in the (usual) partial order

ee) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_3+\lambda_4)) \cong 0$.

The only simple G-module type that could appear as a composition factor is $L(2\lambda_2)$

i) $\operatorname{Hom}_G(L(2\lambda_2), \operatorname{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_3+\lambda_4)))$

$$\cong \operatorname{Ext}^1_G(L(\lambda_3+2\lambda_2), L(\lambda_3+\lambda_4)) \cong 0,$$

by considering the Weyl module $V(2\lambda_2+\lambda_3)$.

ii) $\operatorname{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4)) \cong 0$.

The only simple G-module types that could appear as composition factors are $L(2\lambda_1)$, $L(2[\lambda_2+\lambda_4])$

i) $\operatorname{Hom}_G(L(2\lambda_1), \operatorname{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4)))$

$$\cong \operatorname{Ext}^1_G(L(\lambda_1+\lambda_4+2\lambda_1), L(\lambda_3+\lambda_4)) \cong k,$$

by considering the Weyl module $V(3\lambda_1+\lambda_4)$

ii) $\operatorname{Hom}_G(L(2[\lambda_2+\lambda_4]), \operatorname{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4)))$

$$\preceq \operatorname{Hom}_G(L(2[\lambda_2+\lambda_4]), \operatorname{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4)) \otimes L(2\lambda_1))$$

(as $\operatorname{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_2+\lambda_4) \otimes L(\lambda_1)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\lambda_1+\lambda_4+2[\lambda_2+\lambda_4]), L(\lambda_3+\lambda_4+2\lambda_1)) \cong 0,$$

by considering the Weyl module $V(\lambda_1+2\lambda_2+3\lambda_4)$

We have thus shown that

$$\operatorname{soc}_G(\operatorname{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4))) \cong L(2\lambda_1).$$

Now, the only type of simple G module which can appear as a composition factor of $\operatorname{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4))$, and which extends $L(2\lambda_1)$, is $L(2[\lambda_2+\lambda_4])$. However,

the argument of part (ii) shows that $L(2[\lambda_2 + \lambda_4])$ cannot be a summand of the second socle layer because $L(\lambda_3 + \lambda_4)$ does not extend any of the composition factors of $L(\lambda_1) \otimes L(\lambda_1)$

gg) $\mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_3 + \lambda_4)) \cong 0$

The only simple G-module types that could appear as composition factors are $L(2\lambda_2)$, $L(2[\lambda_3 + \lambda_4])$

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_3 + \lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_3 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + \lambda_2 + 2\lambda_1), L(\lambda_3 + \lambda_4 + 2\lambda_4)) \cong 0,$$

since $3\lambda_1 + \lambda_2$ is not comparable with $\lambda_3 + 3\lambda_4$ in the (usual) partial order

ii) $\mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_3 + \lambda_4)))$

$$\preceq \mathrm{Hom}_G(L(2[\lambda_1 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_3 + \lambda_4)) \otimes L(2\lambda_3))$$

(as $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_4]), L(\lambda_3 + \lambda_4) \otimes L(\lambda_3)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + \lambda_2 + 2[\lambda_1 + \lambda_4]), L(\lambda_3 + \lambda_4 + 2\lambda_3)) \cong 0,$$

since $3\lambda_1 + \lambda_2 + 2\lambda_4$ is not comparable with $3\lambda_3 + \lambda_4$ in the (usual) partial order

hh) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_2 + \lambda_4)) \cong 0$

From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \operatorname{ind}_B^G(H^1(B_1, \lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_1 + \lambda_4))^{(2^{-1})} \cong 0$. However, $H^0(\lambda_1 + \lambda_4) \cong L(\lambda_1 + \lambda_4)$.

ii) $\operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1 + \lambda_4)) \cong 0$

The only simple G-module types that could appear as composition factors are $L(2\lambda_2)$, $L(2[\lambda_3 + \lambda_4])$

i) $\operatorname{Hom}_G(L(2\lambda_2), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1 + \lambda_4)))$

$$\precsim \operatorname{Hom}_G(L(2\lambda_3), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1 + \lambda_4)) \otimes L(2\lambda_1))$$

(as $\operatorname{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\lambda_1 + 2\lambda_3), L(\lambda_1 + \lambda_4 + 2\lambda_1)) \cong 0,$$

since $\lambda_1 + 2\lambda_3$ is not comparable with $3\lambda_1 + \lambda_4$ in the (usual) partial order

ii) $\operatorname{Hom}_G(L(2[\lambda_3 + \lambda_4]), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1 + \lambda_4)))$

$$\precsim \operatorname{Hom}_G(L(2\lambda_3), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1 + \lambda_4)) \otimes L(2\lambda_1))$$

(as $\operatorname{Hom}_G(L(\lambda_3), L(\lambda_3 + \lambda_4) \otimes L(\lambda_1)) \cong \operatorname{Hom}_G(L(\lambda_1 + \lambda_2), L(\lambda_2) \otimes L(\lambda_1)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\lambda_1 + 2\lambda_3), L(\lambda_1 + \lambda_4 + 2\lambda_1)) \cong 0$$

since $\lambda_1 + 2\lambda_3$ is not comparable with $3\lambda_1 + \lambda_4$ in the (usual) partial order

iii) $\operatorname{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)) \cong 0$

The only simple G-module types that could appear as composition factors are $L(2\lambda_4)$, $L(2[\lambda_1 + \lambda_3])$

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)))$

$$\cong \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2\lambda_3), L(\lambda_1 + \lambda_4 + 2\lambda_4)) \cong 0,$$

since $\lambda_2 + 2\lambda_3$ is not comparable with $\lambda_1 + 3\lambda_4$ in the (usual) partial order.

ii) $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)))$

$$\cong \mathrm{Hom}_G(L(2[\lambda_1 + \lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_2]), L(\lambda_1 + \lambda_3) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2[\lambda_1 + \lambda_2]), L(\lambda_1 + \lambda_4 + 2\lambda_4)) \cong 0,$$

since $2\lambda_1 + 3\lambda_2$ is not comparable with $\lambda_1 + 3\lambda_4$ in the (usual) partial order.

kk) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_2)) \cong L(2\lambda_1)$

From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1, \lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_2))^{(2^{-1})} \cong L(\lambda_1)$. However, $H^0(\lambda_2) \cong L(\lambda_2)$

ll) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \cong 0$.

The only simple G-module types that could appear as composition factors are $L(2\lambda_3)$, $L(4\lambda_4)$, $L(2[\lambda_1 + \lambda_2])$, and $L(2[\lambda_1 + \lambda_3 + \lambda_4])$

i) $\operatorname{Hom}_G(L(2\lambda_3), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)))$

$$\cong \operatorname{Hom}_G(L(2\lambda_3), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \otimes L(2\lambda_4))$$

(as $\operatorname{Hom}_G(L(\lambda_2), L(\lambda_3) \otimes L(\lambda_4)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\lambda_1 + 2\lambda_3), L(\lambda_2 + 2\lambda_4)) \cong 0,$$

since $\lambda_1 + 2\lambda_3$ is not comparable with $\lambda_2 + 2\lambda_4$ in the (usual) partial order

ii) $\operatorname{Hom}_G(L(4\lambda_3), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)))$

$$\cong \operatorname{Hom}_G(L(4\lambda_3), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \otimes L(4\lambda_4))$$

(as $\operatorname{Hom}_G(L(\lambda_3), L(\lambda_4) \otimes L(\lambda_4)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\lambda_1 + 4\lambda_3), L(\lambda_2 + 4\lambda_4)) \cong 0,$$

since $\lambda_1 + 4\lambda_3$ is not comparable with $\lambda_2 + 4\lambda_4$ in the (usual) partial order

iii) $\operatorname{Hom}_G(L(2[\lambda_1 + \lambda_2]), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)))$

$$\cong \operatorname{Hom}_G(L(2\lambda_2), \operatorname{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \otimes L(2\lambda_4))$$

(as $\operatorname{Hom}_G(L(\lambda_2), L(\lambda_1 + \lambda_2) \otimes L(\lambda_4)) \cong \operatorname{Hom}_G(L(\lambda_2 + \lambda_4), L(\lambda_3) \otimes L(\lambda_4)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\lambda_1 + 2\lambda_2), L(\lambda_2 + 2\lambda_4)) \cong 0,$$

since $\lambda_1 + 2\lambda_2$ is not comparable with $\lambda_2 + 2\lambda_4$ in the (usual) partial order

iv) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_3+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)))$

$$\preceq \mathrm{Hom}_G(L(2[\lambda_1+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_3+\lambda_4) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2[\lambda_1+\lambda_3]), L(\lambda_2+2\lambda_1)) \cong 0,$$

by considering the Weyl module $V(3\lambda_1+2\lambda_3)$ (We have that $2\lambda_1+\lambda_2 \preceq 3\lambda_1+2\lambda_3$ but $V(3\lambda_1+2\lambda_3)$ has no composition factor isomorphic to $L(2\lambda_1+\lambda_2)$.)

mm) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)) \cong L(2\lambda_4)$.

The only simple G-module types that could appear as composition factors are $L(2\lambda_4)$, $L(2[\lambda_1+\lambda_3])$, and $L(4\lambda_2)$

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4+2\lambda_4), L(\lambda_2)) \cong k,$$

by considering the Weyl module $V(3\lambda_4)$.

ii) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2))) \cong 0$

We have that

$$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)) \otimes L(2\lambda_3))$$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4+2\lambda_2), L(\lambda_2+2\lambda_3)) \cong k,$$

by considering the Weyl module $V(\lambda_2+2\lambda_3)$. However,

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_4) \otimes L(\lambda_3)) \cong k,$$

so that if there were any summands of the form $L(2[\lambda_1 + \lambda_3])$ in the socle of $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2))$, we would have that

$$\dim_k(\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)) \otimes L(2\lambda_3))) \geq 2,$$

because of part (i), as

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1 + \lambda_3) \otimes L(\lambda_3)) \cong k,$$

and

$$\mathrm{Hom}_G(L(\lambda_2), L(\lambda_4) \otimes L(\lambda_3)) \cong k.$$

iii) $\mathrm{Hom}_G(L(4\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)))$

$$\simeq \mathrm{Hom}_G(L(4\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)) \otimes L(4\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 4\lambda_3), L(\lambda_2 + 4\lambda_1)) \cong 0,$$

since $4\lambda_3 + \lambda_4$ is not comparable with $4\lambda_1 + \lambda_2$ in the (usual) partial order.

We have thus shown that

$$\mathrm{soc}_G(\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2))) \cong L(2\lambda_4)$$

Now, the only type of simple G-module which can appear as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2))$, and which extends $L(2\lambda_4)$, is $L(2[\lambda_1 + \lambda_3])$. However, the

argument of part (ii) shows that $L(2[\lambda_1 + \lambda_3])$ cannot be a summand of the second socle layer because $L(\lambda_2)$ does not extend any of the composition factors of $L(\lambda_4) \otimes L(\lambda_3)$

iii) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)) \cong 0$

The only simple G module types that could appear as composition factors are $L(2\lambda_2)$, $L(4\lambda_1)$, $L(2[\lambda_3 + \lambda_4])$, and $L(2[\lambda_1 + \lambda_2 + \lambda_4])$

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k$.)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3 + 2\lambda_3), L(\lambda_2 + 2\lambda_1)) \cong 0,$$

since $3\lambda_3$ is not comparable with $2\lambda_1 + \lambda_2$ in the (usual) partial order

ii) $\mathrm{Hom}_G(L(4\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)))$

$$\precsim \mathrm{Hom}_G(L(4\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)) \otimes L(4\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_1)) \cong k$.)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3 + 4\lambda_3), L(\lambda_2 + 4\lambda_1)) \cong 0,$$

since $4\lambda_2 + \lambda_3$ is not comparable with $4\lambda_1 + \lambda_2$ in the (usual) partial order

iii) $\mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_3+\lambda_4)\otimes L(\lambda_1)) \cong \mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_2)\otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+2\lambda_3), L(\lambda_2+2\lambda_1)) \cong 0,$$

since $3\lambda_3$ is not comparable with $2\lambda_1+\lambda_2$ in the (usual) partial order.

iv) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_2+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)))$

$$\preceq \mathrm{Hom}_G(L(2[\lambda_2+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2))\otimes L(2\lambda_3))$$

(as $\mathrm{Hom}_G(L(\lambda_2+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)\otimes L(\lambda_3)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+2[\lambda_2+\lambda_3]), L(\lambda_2+2\lambda_3)) \cong 0,$$

by considering the Weyl module $V(2\lambda_2+3\lambda_3)$ (We have that $\lambda_2+2\lambda_3 \preceq 2\lambda_2+3\lambda_3$, but $V(2\lambda_2+3\lambda_3)$ has no composition factor isomorphic to $L(\lambda_2+2\lambda_3)$.)

oo) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_1)) \cong 0$

From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong ind_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_1))^{(2^{-1})} \cong 0$. However, $H^0(\lambda_1) \cong L(\lambda_1)$

pp) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)) \cong 0$

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1)\otimes L(\lambda_1)) \cong H^1_{G_1}(L(\lambda_1)\otimes L(\lambda_1)).$$

It follows easily from the long exact sequence in cohomology that the set of composition factors (with multiplicities) of $H^1_{G_1}(M)$ (for any G-module M) is a subset of

the union of the sets of composition factors of $H^1_{G_1}(L)$ as L ranges over the composition factors of M. Therefore, the only simple G-module type that could appear as a composition factor of $H^1_{G_1}(L(\lambda_1) \otimes L(\lambda_1))$ is $L(2\lambda_1)$.

i) $\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)))$

$$\lesssim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2\lambda_2), L(\lambda_1 + 2\lambda_1)) \cong 0,$$

since $2\lambda_2 + \lambda_4$ is not comparable with $3\lambda_1$ in the (usual) partial order

§4.2 Extensions for $(D_4)_1$

We now compute, using the same method as in the preceding section, all of the quantities $\mathrm{Ext}^1_{D_1}(L, M)$, where L and M are restricted simple modules. Utilizing Lemma 3 1 1 together with Table 2-1 and exploiting the symmetry of D_4, we need to list only 9 such computations. We also use the fact that for $G = A_4$, $\mathrm{Ext}^1_{G_1}(L, L) = 0$ for any restricted simple module L

a) $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)) \cong k$

By Lemma 3 1 1 and Table 2-1, the only simple D modules that could appear as composition factors of $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4))$ are $k, L(2\delta_1), L(2[\delta_3 + \delta_4])$, and $L(2\delta_2)$

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)) \cong k$$

by the structure of the Weyl module $V(\delta_2 + \delta_3 + \delta_4)$ (See Table 5 1)

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2\delta_1), L(\delta_2+\delta_3+\delta_4)) \cong 0$$

as $3\delta_1$ is incomparable with $\delta_2+\delta_3+\delta_4$ (in the usual partial order.)

iii) $\mathrm{Hom}_D(L(2[\delta_3+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_2+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4)) \otimes L(2\delta_3))$$

(as $\mathrm{Hom}_D(L(\delta_2+\delta_4), L(\delta_2) \otimes L(\delta_3+\delta_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_D(L(\delta_1+2[\delta_2+\delta_4]), L(\delta_2+\delta_3+\delta_4+2\delta_3)) \cong 0,$$

as $\delta_1+2[\delta_2+\delta_4]$ is incomparable with $\delta_2+\delta_3+\delta_4+2\delta_3$

iv) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_1+\delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4)) \otimes L(2[\delta_3+\delta_4]))$$

(as $\mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_2) \otimes L(\delta_3+\delta_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_D(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_2+\delta_3+\delta_4+2[\delta_3+\delta_4])) \cong 0$$

as $\delta_1+2[\delta_1+\delta_2]$ is incomparable with $\delta_2+\delta_3+\delta_4+2[\delta_3+\delta_4]$

Thus, we have $\mathrm{soc}(\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4))) \cong k$. Now, the only simple D-modules that could appear in the second socle layer are isomorphs of $L(2\delta_2)$ (since $\mathrm{Ext}^1_D(k,k)$, $\mathrm{Ext}^1_D(L(\delta_1),k)$, and $\mathrm{Ext}^1_D(L(\delta_2+\delta_4),k)$ are all zero), however, the argument above shows that

$$\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4))/\mathrm{soc}(\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4))) \cong 0,$$

because $\mathrm{Ext}^1_D(L(\delta_3+\delta_4), L(\delta_1+\delta_2)) \cong 0$

b) $\mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)) \cong 0$

The only simple D modules that can appear as composition factors are $k, L(2\delta_1)$, and $L(2\delta_2)$

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_D(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)) \cong 0,$$

by the structure of the Weyl module

$$V(\delta_2+\delta_3+\delta_4) \subseteq V(\delta_2+\delta_3) \otimes V(\delta_4) \cong L(\delta_3+\delta_2) \otimes L(\delta_4)$$

$$\mathrm{Hom}_D(L(\delta_1), L(\delta_2+\delta_3) \otimes L(\delta_4)) \cong \mathrm{Hom}_D(L(\delta_2+\delta_3), L(\delta_1) \otimes L(\delta_4)) \cong 0,$$

so that the unique composition factor of $V(\delta_2+\delta_3+\delta_4)$ isomorphic to $L(\delta_3+\delta_4)$ must lie in $\mathrm{Rad}^2(V(\delta_2+\delta_3+\delta_4))$, i.e.,

$V(\delta_2+\delta_3+\delta_4) \cong L(\delta_2+\delta_3+\delta_4)//L(\delta_1)//L(\delta_3+\delta_4)$ is uniserial

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_D(L(\delta_3+\delta_4+2\delta_1), L(\delta_2+\delta_3+\delta_4)) \cong 0,$$

by the structure of the Weyl module $V(\delta_3+\delta_4+2\delta_1) \subseteq V(\delta_3+2\delta_1) \otimes V(\delta_4)$, which has a filtration $(L(\delta_3) \otimes L(\delta_4)) \otimes L(2\delta_1)$

$L(\delta_2+\delta_3) \otimes L(\delta_4)$

The multiplicity of $L(\delta_2+\delta_3+\delta_4)$ as a composition factor of this filtration is easily checked to be equal to one, while the multiplicity of $L(\delta_1)$ is 2. On the other hand, we have $V(\delta_2+\delta_3+\delta_4) \subseteq V(\delta_2+\delta_3) \otimes V(\delta_4) \cong L(\delta_2+\delta_3) \otimes L(\delta_4)$. As $L(\delta_2+\delta_3+\delta_4)$ is indeed a composition factor of $V(\delta_3+\delta_4+2\delta_1)$, and since its multiplicity in $V(\delta_3+2\delta_1) \otimes V(\delta_4)$ is one, we must have $V(\delta_2+\delta_3+\delta_4) \subseteq V(\delta_2+\delta_4+2\delta_1)$. Since the multiplicity of $L(\delta_1)$ in $V(\delta_3+\delta_4+2\delta_1)$ is exactly two, we obtain that the first Jantzen layer of $V(\delta_3+\delta_4+2\delta_1)$ (which consists of 2 composition factors of $L(\delta_1)$, and one factor of $L(\delta_2+\delta_3+\delta_4)$,) must be uniserial. Otherwise, there would exist a submodule M of $V(\delta_3+\delta_4+2\delta_1)$ containing a composition factor isomorphic to $L(\delta_2+\delta_3+\delta_4)$ and no composition factors isomorphic to $L(\delta_1)$. The image of $V(\delta_2+\delta_3+\delta_4)$ under the canonical map to $V(\delta_3+\delta_4+2\delta_1)/M$ would then be zero because the multiplicity of $L(\delta_2+\delta_3+\delta_4)$ as a composition factor of $V(\delta_3+\delta_4+2\delta_1)$ is one. Thus we would have $V(\delta_2+\delta_3+\delta_4) \subseteq M$, which is absurd.

iii) $\operatorname{Hom}_D(L(2\delta_2), \operatorname{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)))$

$$\cong \operatorname{Ext}^1_D(L(\delta_3+\delta_4+2\delta_2), L(\delta_2+\delta_3+\delta_4)) \cong 0$$

We consider the Weyl module $V(\delta_3+\delta_4+2\delta_2)$. The multiplicity of $L(\delta_2+\delta_3+\delta_4)$ as a composition factor is equal to one, as is that of $L(\delta_1+\delta_2+2\delta_3)$. We also have

$$V(\delta_3+\delta_4+2\delta_2) \subseteq V(\delta_3) \otimes V(\delta_4+2\delta_2) \cong L(\delta_3) \otimes V(\delta_4+2\delta_2)$$

A quick check of the filtration factors of $L(\delta_3) \otimes V(\delta_4+2\delta_2)$ that result from the composition factors of $V(\delta_4+2\delta_2)$ shows the multiplicities of $L(\delta_2+\delta_3+\delta_4)$ and

$L(\delta_1+\delta_2+2\delta_3)$ to be 4 and 1 respectively, all occurring inside of a submodule which is isomorphic to $L(\delta_1+\delta_2+\delta_3)\otimes L(\delta_3)$. We also have

$$V(\delta_1+\delta_2+2\delta_3)\subseteq V(\delta_1+\delta_2+\delta_3)\otimes V(\delta_3)\cong V(\delta_1+\delta_2+\delta_3)\otimes L(\delta_3),$$

which has $L(\delta_1+\delta_2+\delta_3)\otimes L(\delta_3)$ as a homomorphic image. Let M be the image of $V(\delta_1+\delta_2+2\delta_3)$ under this homomorphism. In particular, $M\subseteq V(\delta_3)\otimes V(\delta_4+2\delta_2)$. We note that $M\not\cong 0$, since the kernel contains no composition factors isomorphic to $L(\delta_1+\delta_2+2\delta_3)$. Also, M has a composition factor isomorphic to $L(\delta_2+\delta_2+\delta_4)$ (since the kernel contains no such composition factor.) Because the multiplicity of $L(\delta_1+\delta_2+2\delta_3)$ as a composition factor of $V(\delta_3)\otimes V(\delta_4+2\delta_2)$ is one, we have that

$$M\subseteq V(\delta_3+\delta_4+2\delta_2)$$

This implies that the unique composition factor of $V(\delta_3+\delta_4+2\delta_2)$ isomorphic to $L(\delta_2+\delta_3+\delta_4)$ must occur inside $\mathrm{Rad}^2(V(\delta_2+\delta_4+2\delta_2))$.

c) $\mathrm{Ext}^1_{D_1}(L(\delta_1+\delta_2),L(\delta_2+\delta_3+\delta_4))\cong 0$

The only candidates for summands of the socle are $L(2\delta_3)$ and $L(2\delta_4)$.

$$\mathrm{Hom}_D(L(2\delta_2),\mathrm{Ext}^1_{D_1}(L(\delta_1+\delta_2),L(\delta_2+\delta_3+\delta_4)))$$

$$\cong\mathrm{Hom}_D(L(2[\delta_2+\delta_4]),\mathrm{Ext}^1_{D_1}(L(\delta_1+\delta_2),L(\delta_2+\delta_3+\delta_4))\otimes L(2[\delta_3+\delta_4])),$$

(as $\mathrm{Hom}_D(L(\delta_2+\delta_4),L(\delta_3)\otimes L(\delta_2+\delta_4))\cong k$,)

$$\cong\mathrm{Ext}^1_D(L(\delta_1+\delta_2+2[\delta_2+\delta_4]),L(\delta_2+\delta_2+\delta_4+2[\delta_3+\delta_4]))\cong 0,$$

as $\delta_1 + \delta_2 + 2[\delta_2 + \delta_4]$ is incomparable with $\delta_2 + \delta_3 + \delta_4 + 2[\delta_3 + \delta_4]$ in the usual partial order. ($L(2\delta_4)$ is handled similarly.)

d) $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)) \cong L(2\delta_3) \oplus L(2\delta_4)$

The only possible isomorphism types of summands of the socle are $k, L(2\delta_2), L(2\delta_3), L(2\delta_4), L(2\delta_2), L(2[\delta_3+\delta_4]), L(2[\delta_1+\delta_4]), L(2[\delta_1+\delta_3]), L(2[\delta_2+\delta_3])$, and $L(2[\delta_2+\delta_4])$

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1), L(\delta_1 + \delta_2)) \cong 0,$$

by considering $V(\delta_1 + \delta_2) \cong L(\delta_1 + \delta_2)$

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1 + 2\delta_1), L(\delta_1 + \delta_2)) \cong 0,$$

by considering $V(3\delta_1)$.

iii) $\mathrm{Hom}_D(L(2\delta_3), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1 + 2\delta_3), L(\delta_1 + \delta_2)) \cong k,$$

by considering $V(\delta_1 + 2\delta_3)$. (Similarly for $L(2\delta_4)$.)

iv) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)))$

$$\cong \operatorname{Ext}^1_G(L(\delta_1 + 2\delta_2), L(\delta_1 + \delta_2)) \cong 0,$$

by considering $V(\delta_1 + 2\delta_2)$

v) $\operatorname{Hom}_D(L(2[\delta_3 + \delta_4]), \operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)))$

$$\cong \operatorname{Ext}^1_G(L(\delta_1 + 2[\delta_3 + \delta_4]), L(\delta_1 + \delta_2)) \cong 0,$$

by considering $V(\delta_1 + 2[\delta_3 + \delta_4])$

vi) $\operatorname{Hom}_D(L(2[\delta_1 + \delta_4]), \operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)))$

$$\precsim \operatorname{Hom}_D(L(2[\delta_1 + \delta_2]), \operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)) \otimes L(2\delta_4)),$$

(as $\operatorname{Hom}_D(L(\delta_1 + \delta_2), L(\delta_4) \otimes L(\delta_1 + \delta_4)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\delta_1 + 2[\delta_1 + \delta_2]), L(\delta_1 + \delta_2 + 2\delta_4)) \cong 0,$$

by considering $V(\delta_1 + 2[\delta_1 + \delta_2])$ (Similarly for $L(2[\delta_1 + \delta_3])$)

vii) $\operatorname{Hom}_D(L(2[\delta_2 + \delta_2]), \operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)))$

$$\precsim \operatorname{Hom}_D(L(2[\delta_1 + \delta_2]), \operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2)) \otimes L(2\delta_4)),$$

(as $\operatorname{Hom}_D(L(\delta_1 + \delta_2), L(\delta_4) \otimes L(\delta_2 + \delta_3)) \cong k$,)

$$\cong \operatorname{Ext}^1_G(L(\delta_1 + 2[\delta_1 + \delta_2]), L(\delta_1 + \delta_2 + 2\delta_4)) \cong 0,$$

by considering $V(\delta_1 + 2[\delta_1 + \delta_2])$ (Similarly for $L(2[\delta_2 + \delta_4])$.)

We have thus shown that

$$\operatorname{soc}_D(\operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2))) \cong L(2\delta_3) \oplus L(2\delta_4)$$

Now, the only types of simple D-modules which can appear as composition factors of $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2))$, that extend $L(2\delta_3)$ or $L(2\delta_4)$, are $L(2[\delta_1+\delta_4])$ and $L(2[\delta_1+\delta_3])$. However, the argument of part (vi) shows that $L(2[\delta_1+\delta_4])$ (similarly for $L(2[\delta_1+\delta_3])$) cannot be a summand of the second socle layer because $L(\delta_1+\delta_2)$ does not extend any of the composition factors of $L(\delta_4)\otimes[L(\delta_3)\oplus L(\delta_4)]$.

e) $\mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)) \cong 0$

The only possible isomorphism types that can appear in the socle are k, $L(2\delta_1)$, $L(2\delta_3)$, $L(2\delta_4)$, $L(2\delta_2)$, $L(2[\delta_1+\delta_4])$, and $L(2[\delta_1+\delta_3])$.

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4), L(\delta_1+\delta_2)) \cong 0,$$

by considering $V(\delta_1+\delta_2)\cong L(\delta_1+\delta_2)$.

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4+2\delta_1), L(\delta_1+\delta_2)) \cong 0,$$

by considering $V(\delta_3+\delta_4+2\delta_1)$

iii) $\mathrm{Hom}_D(L(2\delta_3), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4+2\delta_3), L(\delta_1+\delta_2)) \cong 0,$$

by considering $V(\delta_3+\delta_4+2\delta_3)$. We have

$$V(3\delta_3+\delta_4) \subseteq V(2\delta_3+\delta_4)\otimes V(\delta_3) \cong V(2\delta_3+\delta_4)\otimes L(\delta_3),$$

which has a filtration $(L(\delta_3) \otimes L(\delta_4)) \otimes L(2\delta_3)$

$$L(\delta_3) \otimes L(\delta_2 + \delta_4).$$

Now,

$$\mathrm{Hom}_D(L(\delta_1 + 2\delta_4), L(\delta_3) \otimes L(\delta_2 + \delta_4)) \cong \mathrm{Hom}_D(L(\delta_2 + \delta_4 + 2\delta_4), L(\delta_1) \otimes L(\delta_3)) \cong 0,$$

while $L(\delta_3) \otimes L(\delta_4)) \otimes L(2\delta_3)$ has no composition factor isomorphic to $L(\delta_1 + 2\delta_4)$. Thus $\mathrm{Hom}_D(L(\delta_1 + 2\delta_4), V(\delta_3 + \delta_4 + 2\delta_3)) \cong 0$, which implies that $\mathrm{Rad}^2(V(\delta_3 + \delta_4 + 2\delta_3)) \cong L(\delta_1 + \delta_2)$

iv) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_3 + \delta_4), L(\delta_1 + \delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3 + \delta_4 + 2\delta_2), L(\delta_1 + \delta_2)) \cong 0,$$

by considering $V(\delta_3 + \delta_4 + 2\delta_2)$

v) $\mathrm{Hom}_D(L(2[\delta_1 + \delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_3 + \delta_4), L(\delta_1 + \delta_2)))$

$$\cong \mathrm{Hom}_D(L(2[\delta_2 + \delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_3 + \delta_4), L(\delta_1 + \delta_2)) \otimes L(2[\delta_1 + \delta_2]),$$

(since $\mathrm{Hom}_D(L(\delta_2 + \delta_4), L(\delta_1 + \delta_4) \otimes L(\delta_1 + \delta_2)) \not\cong 0$, by Lemma 3 4 3)

$$\cong \mathrm{Ext}^1_G(L(\delta_3 + \delta_4 + 2[\delta_2 + \delta_4]), L(\delta_1 + \delta_2 + 2[\delta_1 + \delta_2])) \cong 0,$$

as $\delta_3 + \delta_4 + 2[\delta_2 + \delta_4]$ is incomparable (in the usual partial order) with $\delta_1 + \delta_2 + 2[\delta_1 + \delta_2]$

f) $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3 + \delta_4)) \cong k \oplus L(2\delta_1)$

The socle candidates are $k, L(2\delta_1), L(2\delta_3), L(2\delta_4), L(2\delta_2), L(4\delta_1), L(4\delta_3), L(4\delta_4)$, $L(2[\delta_3 + \delta_4]), L(2[\delta_1 + \delta_4]), L(2[\delta_1 + \delta_3]), L(2[\delta_1 + \delta_2]), L(2[\delta_2 + \delta_3]), L(2[\delta_2 + \delta_4])$, and $L(2[\delta_1 + \delta_2 + \delta_4])$.

i) $\mathbf{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathbf{Ext}^1_G(L(\delta_1), L(\delta_3+\delta_4)) \cong k,$$

by considering $V(\delta_3+\delta_4)$

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(3\delta_1), L(\delta_3+\delta_4)) \cong k,$$

by considering $V(3\delta_1)$

iii) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2\delta_2), L(\delta_3+\delta_4)) \cong 0,$$

by considering $V(\delta_1+2\delta_2)$

iv) $\mathrm{Hom}_D(L(2\delta_3), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathbf{Hom}_D(L(2[\delta_1+\delta_2]), \mathbf{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(2\delta_1)),$$

(as $L(\delta_2) \otimes L(\delta_1) \cong L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$)

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_3+\delta_4+2\delta_1)) \cong 0,$$

by considering $V(\delta_1+2[\delta_1+\delta_2])$

v) $\mathrm{Hom}_D(L(4\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+4\delta_1), L(\delta_3+\delta_4)) \cong 0,$$

by considering $V(5\delta_1)$

vi) $\mathrm{Hom}_D(L(4\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\precsim \mathrm{Hom}_D(L(4[\delta_1+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(4\delta_2)),$$

(as $L(\delta_2) \otimes L(\delta_3) \cong L(\delta_1+\delta_4) \oplus L(\delta_2+\delta_3)$.)

$$\cong \mathrm{Ext}^1_G(L(\delta_1+4[\delta_1+\delta_4]), L(\delta_3+\delta_4+4\delta_2)) \cong 0,$$

as $5\delta_1+4\delta_4$ is not comparable with $4\delta_2+\delta_3+\delta_4$ in the (usual) partial order.

vii) $\mathrm{Hom}_D(L(2[\delta_3+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\precsim \mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(2\delta_1)),$$

(as $\mathrm{Hom}_D(L(\delta_2), L(\delta_1) \otimes L(\delta_3+\delta_4)) \cong \mathrm{Hom}_D(L(\delta_3+\delta_4), L(\delta_1) \otimes L(\delta_2)) \cong k$)

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2\delta_2), L(\delta_3+\delta_4+2\delta_1)) \cong 0,$$

by considering $V(\delta_1+2\delta_2)$. Suppose the first Jantzen layer is semisimple. Since $\mathrm{Ext}^1_D(L(\delta_2+\delta_3+\delta_4), L(\delta_1)) \cong k$, (and $\mathrm{Ext}^1_D(L(\delta_1), L(\delta_1)) \cong 0$,) we would have

$$\dim_k(\mathrm{Hom}_D(L(\delta_1), V(\delta_1+2\delta_2))) \geq 2.$$

However, $V(\delta_1+2\delta_2) \subseteq V(\delta_1+\delta_2) \otimes V(\delta_2) \cong L(\delta_1+\delta_2) \otimes V(\delta_2)$, which has a filtration

$L(\delta_1+\delta_2) \otimes L(\delta_2)$

$2L(\delta_1+\delta_2)$,

whence $V(\delta_1+2\delta_2) \subseteq L(\delta_1+\delta_2) \otimes L(\delta_2)$, but

$$\mathrm{Hom}_D(L(\delta_1), L(\delta_1+\delta_2) \otimes L(\delta_2)) \cong \mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_1) \otimes L(\delta_2)) \cong k$$

Therefore, (since Jantzen layers are self dual), the first Jantzen layer must be unserial, which proves

$$\mathrm{Ext}^1_D(L(\delta_1+2\delta_2), L(\delta_3+\delta_4+2\delta_1)) \cong 0.$$

vui) $\mathrm{Hom}_D(L(2[\delta_1+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_D(L(\delta_1+2[\delta_1+\delta_4]), L(\delta_3+\delta_4)) \cong 0,$$

by considering $V(3\delta_1+2\delta_4)$.

ix) $\mathrm{Hom}_D(L(2[\delta_1+\delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\preceq \mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(2\delta_1)),$$

(as $\mathrm{Hom}_D(L(\delta_2), L(\delta_1) \otimes L(\delta_1+\delta_2)) \cong \mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_1) \otimes L(\delta_2)) \cong k$)

$$\cong \mathrm{Ext}^1_D(L(\delta_1+2\delta_2), L(\delta_3+\delta_4+2\delta_1)) \cong 0,$$

as in (vii) above.

x) $\mathrm{Hom}_D(L(2[\delta_2+\delta_3]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_4)))$

$$\preceq \mathrm{Hom}_D(L(2[\delta_1+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(2\delta_2)),$$

(as $\mathrm{Hom}_D(L(\delta_1+\delta_4), L(\delta_2) \otimes L(\delta_2+\delta_3)) \cong \mathrm{Hom}_D(L(\delta_2+\delta_3), L(\delta_2) \otimes L(\delta_1+\delta_4)) \cong k$)

$$\cong \mathrm{Ext}^1_D(L(\delta_1+2[\delta_1+\delta_4]), L(\delta_3+\delta_4+2\delta_2)) \cong 0,$$

as $3\delta_1+2\delta_4$ is not comparable with $2\delta_2+\delta_3+\delta_4$ in the (usual) partial order

xi) $\operatorname{Hom}_D(L(2[\delta_1+\delta_3+\delta_4]), \operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \operatorname{Hom}_D(L(2[\delta_1+\delta_2+\delta_4]), \operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(2[\delta_2+\delta_3])),$$

(as $\operatorname{Hom}_D(L(\delta_1+\delta_2+\delta_4), L(\delta_1+\delta_3+\delta_4) \otimes L(\delta_2+\delta_3)) \not\cong 0$, by Lemma 3 4 4)

$$\cong \operatorname{Ext}^1_G(L(\delta_1+2[\delta_1+\delta_2+\delta_4]), L(\delta_3+\delta_4+2[\delta_2+\delta_3])) \cong 0,$$

as $3\delta_1+2\delta_2+2\delta_4$ is not comparable with $2\delta_2+3\delta_3+\delta_4$ in the (usual) partial order

We have thus shown

$$\operatorname{soc}_D(\operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4))) \cong k \oplus L(2\delta_1)$$

Therefore, the only candidates for summands of the second socle layer are possible composition factors of $\operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4))$ that extend either k or $L(2\delta_1)$

$$L(2\delta_2), L(2[\delta_1+\delta_3+\delta_4]), L(2[\delta_3+\delta_4])$$

However, the argument of (iv) shows that $L(2\delta_2)$ does not appear in the second socle layer because $L(2[\delta_1+\delta_2])$ does not extend any of the composition factors of $[k \oplus L(2\delta_1)] \otimes L(2\delta_1)$ The argument of (xi) similarly applies to $L(2[\delta_1+\delta_3+\delta_4])$, if we argue more carefully We make the observation that $\operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4))^{(2^{-1})}$ must decompose as a direct sum $L \oplus M$, say, where L contains only composition factors with highest weight in the same linkage class as zero, and where M contains only composition factors with highest weight in the same linkage class as δ_1 (and with $\operatorname{soc}_D(L) \cong k$ and $\operatorname{soc}_D(M) \cong L(\delta_1)$ Now we argue that $L(\delta_1+\delta_2+\delta_4)$ cannot

appear in the second socle layer of L by the argument of (xi) because $L(\delta_1+\delta_2+\delta_4)$ does not extend $k \otimes L(\delta_2+\delta_3) \cong L(\delta_2+\delta_3)$

We must argue a bit differently for $L(2[\delta_3+\delta_4])$. To show that there are no summands isomorphic to $L(2[\delta_3+\delta_4])$ in the second socle layer, it is sufficient to show

$$\dim_k(\mathrm{Hom}_D(L(2[\delta_1+\delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(2\delta_2)))$$

$$= \dim_k(\mathrm{Ext}^1_D(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_3+\delta_4+2\delta_2))) < 2,$$

since

$$\mathrm{Hom}_D(L(2[\delta_1+\delta_2]), [k \oplus L(2\delta_1)] \otimes L(\delta_2)) \cong k,$$

$$\mathrm{Hom}_D(L(2[\delta_1+\delta_2]), L(2[\delta_3+\delta_4]) \otimes L(\delta_2)) \cong k,$$

and $L(2[\delta_1+\delta_2])$ does not extend any of the composition factors of $[k \oplus L(2\delta_1)] \otimes L(\delta_2)$. We consider the Weyl module $V(3\delta_1+2\delta_2])$. Now,

$$\mathrm{Ext}^1_D(L(\delta_3+\delta_4), L(4\delta_1+\delta_3+\delta_4)) \cong \mathrm{Ext}^1_D(k, L(4\delta_1)) \cong \mathrm{Ext}^1_D(k, L(\delta_1)) \cong 0,$$

and

$$\mathrm{Ext}^1_D(L(\delta_1+2[\delta_1+\delta_2]), L(4\delta_1+\delta_3+\delta_4))$$

$$\cong \mathrm{Hom}_D(L(2[\delta_1+\delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \otimes L(4\delta_1)) \cong 0,$$

by considering all possible composition factors of $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4))$. We conclude that

$$\dim_k(\mathrm{Ext}^1_D(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_3+\delta_4+2\delta_2))) \leq 1,$$

since the first Jantzen layer of $V(3\delta_1+2\delta_2)$ consists of $2L(2\delta_2+\delta_3+\delta_4)$, $1L(4\delta_1+\delta_3+\delta_4)$, at most $1L(3\delta_1)$, and an unknown multiplicity of $L(\delta_3+\delta_4)$.

The remaining nonzero $\mathrm{Ext}^1_{D_1}$ modules for D_4 were calculated by Sin [20]

g) $\mathrm{Ext}^1_{D_1}(k, L(\delta_2)) \cong 2k \oplus L(2\delta_1) \oplus L(2\delta_3) \oplus L(2\delta_4)$

h) $\mathrm{Ext}^1_{D_1}(k, L(\delta_1 + \delta_3 + \delta_4)) \cong k \oplus L(2\delta_2)$

i) $\mathrm{Ext}^1_{D_1}(L(\delta_2), L(\delta_1 + \delta_3 + \delta_4)) \cong k$

§4.3 Extensions for $(C_4)_\tau$

From Sin [20] and the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G_\sigma \arrow[r] & G_1 \arrow[r, "\sigma"] & \tilde{G}_\tau \arrow[r] & 1 \\ 1 \arrow[r] & T_\sigma \arrow[r] \arrow[u, "\cup"'] & D_1 \arrow[r, "\sigma"] \arrow[u, "\cup"'] & \tilde{G}_\tau \arrow[r] \arrow[u, equal] & 1 \end{tikzcd}$$

we obtain that

$$H^*(D_1, M) \cong H^*(\tilde{G}_\tau, M)$$

as D- modules for any $\tilde{G}$ module M (i.e., that $H^*(D_1, M) \cong res_D[H^*(\tilde{G}_\tau, M)^{(\sigma)}]$)
By commutativity of the diagram

$$\begin{tikzcd} G \arrow[r, "\sigma"] & \tilde{G} \arrow[r, "\tau"] & G \\ D \arrow[r, "\sigma"] \arrow[u, "\cup"'] & \tilde{D} \arrow[r, "\tau"] \arrow[u, "\cup"'] & D \arrow[u, "\cup"'] \end{tikzcd}$$

and the fact that $\tau \circ \sigma$ is the Frobenius map, we get that

$$H^*(D_1, M)^{(2^{-1})} \cong res_D[H^*(\tilde{G}_\tau, M)^{(\tau^{-1})}]$$

The 5-term sequence that results from the Hochschild Serre spectral sequence for the pair $(\tilde{G}, \tilde{G}_\tau)$ is

$$0 \to \mathrm{Ext}^1_G(L(\lambda), \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\lambda^0), \tilde{L}(\mu^0))^{(\tau^{-1})} \otimes L(\mu)) \to \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu))$$

$$\to \mathrm{Hom}_G(L(\bar{\lambda}), \mathrm{Ext}^1_{\tilde{G}_r}(\tilde{L}(\lambda^0), \tilde{L}(\mu^0))^{(\tau^{-1})} \otimes L(\bar{\mu}))$$

$$\to \mathrm{Ext}^2_G(L(\bar{\lambda}), \mathrm{Hom}_{\tilde{G}_r}(\tilde{L}(\lambda^0), \tilde{L}(\mu^0))^{(\tau^{-1})} \otimes L(\bar{\mu})) \to \mathrm{Ext}^2_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu))$$

We now use these facts, together with the information gathered in §5 1, to calculate the ext groups for $\tilde{G}_r$

Lemma 4.3.1 *The $Ext^1_{\tilde{G}_r}$ s for $\tilde{G} = C_4$ are as follows*

a) $Ext^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_2 + \tilde{\omega}_3))^{(\tau^{-1})} \cong k$

b) $Ext^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_2))^{(\tau^{-1})} \cong L(\omega_4)$

c) $Ext^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_3))^{(\tau^{-1})} \cong H^0(\omega_1)$

d) $Ext^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_2), \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})} \cong k$

e) $Ext^1_{\tilde{G}_r}(k, \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})} \cong k \oplus L(\omega_2)$

f) $Ext^1_{\tilde{G}_r}(k, \tilde{L}(\tilde{\omega}_2))^{(\tau^{-1})} \cong M \oplus L(\omega_4)$,

where M is the unique (up to isomorphism) uniserial G- module with composition series

$$\begin{matrix} k \\ L(\omega_1) \\ k \end{matrix}$$

Proof The right hand sides of (a), (b), and (d) are the unique G- modules that restrict to the appropriate $\mathrm{Ext}^1_{D_r}$s Result (c) follows from consideration of the Weyl module $\tilde{V}(\tilde{\omega}_3)$ (i.e , that $\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_3)) \cong 0$,) together with the five-term sequence. The argument for result (f) is similar, by considering the Weyl modules $\tilde{V}(\tilde{\omega}_2)$ and $\tilde{V}(2\omega_1)$ Finally, to prove (e), we observe that if $E = \mathrm{Ext}^1_{\tilde{G}_r}(k, \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})}$ were uniserial, then it would be the σ twist of a $\tilde{G}$ module (namely, either

$\tilde{H}^0(\bar{\omega}_2)^{(\sigma)}$ or $\tilde{V}(\bar{\omega}_2)^{(\sigma)}$). This is because $\mathrm{Ext}^1_G(k, L(\omega_2)) \cong k$ and thus there is a unique (up to isomorphism) uniserial G-module with composition series

$$\begin{matrix} k & & L(\omega_2) \\ L(\omega_2), & \text{resp.,} & k \end{matrix}$$

We now use the fact that

$$\mathrm{Hom}_{D_1}(k, M^{(\sigma)}) \cong (\mathrm{Hom}_{\tilde{G}_1}(k, M))^{(\sigma)} \quad [\cong \mathrm{Hom}_{G_1}(k, M^{(\sigma)})]$$

as D-modules (for arbitrary $\tilde{G}$ modules M) because σ maps D_1 (as well as G_1) onto $\tilde{G}_\tau$. However, we know that the restriction of E to D is split.

§4.4 Socles of Tensor Products for B_4 and C_4

We need several more lemmas before we can obtain the Ext^1_Gs (and $\mathrm{Ext}^1_{\tilde{G}}$s) for B_4 and C_4.

LEMMA 4.4.1 *Let μ, ν be τ-restricted weights for $\tilde{G} = C_4$. Then*

$$Hom_{\tilde{G}_1}(\tilde{L}(\mu), \tilde{L}(\bar{\omega}_1) \otimes \tilde{L}(\nu))$$

is $\tilde{G}$-trivial.

PROOF Apply the results of Chapter 3 together with the fact that

$$\mathrm{Hom}_{D_1}(k, M^{(\sigma)}) \cong (\mathrm{Hom}_{\tilde{G}_\tau}(k, M))^{(\sigma)}$$

as D-modules (for arbitrary $\tilde{G}$ modules M)

Lemma 4.4.2 *Let M be any composition factor of $E = \mathrm{Ext}^1_{G_\tau}(\bar{L}(\mu^0), \bar{L}(\nu^0))^{(\tau^{-1})}$, where μ^0, ν^0 are any two τ-restricted weights. Then if λ, β is any pair of 2-restricted weights for G, then*

$$Hom_{G_1}(L(\lambda), M \otimes L(\beta))$$

is G trivial.

Proof. This follows for $M = k, L(\omega_1)$, and $L(\omega_2)$, by Lemma 4.3.2, and Lemma 3.4.1, since these are all σ twists of G modules. For these cases, writing $\lambda = \lambda^0 + \sigma\lambda^1$ and $\beta = \beta^0 + \sigma\beta^1$, we have that

$$\mathrm{Hom}_{G_1}(L(\lambda), M \otimes L(\beta))$$

$$\cong [\mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\beta^0)) \otimes Hom(L(\sigma\lambda^1), M \otimes L(\sigma\beta^1))]^{G_1}$$

$$\cong \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\beta^0)) \otimes \mathrm{Hom}_{G_1}(L(\sigma\lambda^1), M \otimes L(\sigma\beta^1)),$$

since $\mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\beta^0))$ is either zero or k.

If $M = L(\omega_4)$, we have in fact that

$$\mathrm{Hom}_{G_1}(L(\lambda), L(\omega_4) \otimes L(\beta))$$

$$\cong [\mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\omega_4) \otimes L(\beta^0)) \otimes Hom(L(\sigma\lambda^1), L(\sigma\beta^1))]^{G_1}$$

$$\cong \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\omega_4) \otimes L(\beta^0)) \otimes \mathrm{Hom}_{G_1}(L(\sigma\lambda^1), L(\sigma\beta^1)),$$

since

$$\mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\omega_4) \otimes L(\beta^0))$$

is either zero or trivial.

Lemma 4.4.3. *Let $E = Ext^1_{\tilde{G}_1}(\tilde{L}(\mu^0), \tilde{L}(\nu^0))^{(\tau^{-1})}$, where μ^0, ν^0 are any two τ-restricted weights. Then if λ, β is any pair of 2-restricted weights for G, then*

$$Hom_{G_1}(L(\lambda), E \otimes L(\beta))$$

is G-trivial.

Proof. The G-composition factors of $\mathrm{Hom}_{G_1}(L(\lambda), E \otimes L(\beta))$ form a subset of the composition factors of $\mathrm{Hom}_{G_1}(L(\lambda), M \otimes L(\beta))$ as M ranges over the composition factors of E. Now use the preceding lemma and the fact that $\mathrm{Ext}^1_G(k,k) = 0$.

Finally, we make the observation that the $\mathrm{Ext}^1_{G_\sigma}$s for $G = B_4$ are completely determined by the fact that $L(\omega_4)$ is injective for G_σ and $\mathrm{Ext}^1_{G_\sigma}(k,k) \cong L(\omega_1)$. The latter remark follows from the well known fact [2, 7] that for $\tilde{G} = C_4$,

$$\mathrm{Ext}^1_{\tilde{G}_1}(k,k)^{(2^{-1})} \cong \tilde{L}(\tilde{\omega}_1)$$

and the five-term sequence for the pair $(\tilde{G}_1, \tilde{G}_\tau)$.

CHAPTER 5

MODULE EXTENSIONS FOR THE ALGEBRAIC GROUPS

§5.1 Final Ext^1_G Computations for the Algebraic Groups

We are now ready to compute the Ext^1_Gs for the simply connected algebraic groups of type A_4, B_4, C_4 and D_4. For the groups of type D_4 and A_4, we have the following result.

PROPOSITION 5.1.1. *Let G be the simply connected algebraic group of type A_4 (resp D_4). Let $\lambda = \sum_i 2^i\lambda^i$ and $\mu = \sum_j 2^j\mu^j$. Then $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$ unless $\lambda - \mu = 2^s(\lambda^s - \mu^s) + 2^{s+1}(\lambda^{s+1} - \mu^{s+1})$ for some $s \geq 0$, in which case the space of extensions can be found from Tables 5-1 to 5-2 (resp Tables 5-3 to 5-6), after applying the reduction (obtained from the 5-term Hochschild-Serre sequence)*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda^s + 2\lambda^{s+1}), L(\mu^s + 2\mu^{s+1}))$$

$$\cong \mathrm{Hom}_G(L(\lambda^{s+1}), \mathrm{Ext}^1_{G_1}(L(\lambda^s), L(\mu^s))^{(2^{-1})} \otimes L(\mu^{s+1}))$$

PROOF. Each of the $E = \mathrm{Ext}^1_{G_1}(L(\lambda^s), L(\mu^s))^{(2^{-1})}$ for these groups have been shown to be semisimple, and for each possible simple summand M, it has been shown that $\mathrm{Hom}_{G_1}(A, M \otimes L(\mu^{s+1}))$ is G trivial (by considering the dimension of the smallest nontrivial G-module) for every simple 2-restricted G-module A. Thus,

$$\mathrm{Hom}_G(L(\lambda^{s+1} + 2\lambda), E \otimes L(\mu^{s+1} + 2\mu))$$

$$\cong [\mathrm{Hom}_{G_1}(L(\lambda^{s+1}), E \otimes L(\mu^{s+1})) \otimes \mathrm{Hom}(L(2\lambda), L(2\mu))]^G$$

$$\cong \mathrm{Hom}_{G_1}(L(\lambda^{s+1}), E \otimes L(\mu^{s+1})) \otimes \mathrm{Hom}_G(L(2\bar{\lambda}), L(2\bar{\mu})),$$

where, for convenience in notation, we have adopted the abbreviations

$$\bar{\lambda} = \sum_{i \geq 0} 2^i \lambda^{s+2+i}, \qquad \bar{\mu} = \sum_{i \geq 0} 2^i \mu^{s+2+i}$$

Proposition 5.1.2 *Let $\tilde{G}$ be the simply connected algebraic group of type C_4. Let $\lambda = \sum_i 2^i(\lambda^{2i} + \tau\lambda^{2i+1})$ and $\mu = \sum_i 2^i(\mu^{2i} + \tau\mu^{2i+1})$. Then $Ext^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu)) = 0$ unless $\lambda - \mu = 2^s(\lambda^{2s} - \mu^{2s} + \tau[\lambda^{2s+1} - \mu^{2s+1}]) + 2^{s+1}(\lambda^{2(s+1)} - \mu^{2(s+1)})$ for some $s \geq 0$, with $\lambda^{2s} \neq \mu^{2s}$, in which case there are 2 different situations to be considered*

a) If $s \geq 1$, $\lambda^{2s-1} = \mu^{2s-1} = 0$, and λ^{2s} is in a different congruence class (with respect to the root lattice) than μ^{2s}, then the extension group is zero unless $\lambda^{2s+1} + \sigma\lambda^{2(s+1)} = \mu^{2s+1} + \sigma\mu^{2(s+1)}$, in which case

$$Ext^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu)) \cong Hom_{\tilde{G}}(\tilde{L}(\lambda^{2s}), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^{2s})),$$

b) In all other cases, i.e. if

i) $s = 0$, or

ii) $s \geq 1$ with $\lambda^{2s-1} = \mu^{2s-1} = \omega_4$, or

iii) $s \geq 1$ with $\lambda^{2s-1} = \mu^{2s-1} = 0$, and λ^{2s} is in the same congruence class than μ^{2s},

then we have

$$Ext^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu))$$
$$\cong Hom_G(L(\lambda^{2s+1} + \sigma\lambda^{2(s+1)}), Ext^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^{2s}), \tilde{L}(\mu^{2s}))^{(\tau^{-1})} \otimes L(\mu^{2s+1} + \sigma\mu^{2(s+1)}))$$

Proof First of all, if the smallest integer k such that $\lambda^k \neq \mu^k$ is odd, say $k = 2j+1$, then upon applying reductions obtained from the 5-term Hochschild Serre sequences

for the pairs $(\tilde{G}, \tilde{G}_\tau)$ and (G, G_σ), (using that $\mathrm{Ext}^1_{\tilde{G}_\tau}(L(\nu^0), L(\nu^0)) = 0$ for any τ-restricted weight ν^0, and that $\mathrm{Ext}^1_{G_\sigma}(k,k)^{(\sigma^{-1})} \cong \tilde{L}(\tilde{\omega}_1)$ is in a different congruence class than zero,) we obtain

$$\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu))$$

$$\cong \mathrm{Ext}^1_G(L(\sum_{i \geq j} 2^{i-j}[\lambda^{2i+1} + \sigma\lambda^{2(i+1)}]), L(\sum_{i \geq j} 2^{i-j}[\mu^{2i+1} + \sigma\mu^{2(i+1)}]))$$

Now, if we apply one last time the 5-term Hochschild-Serre sequence for the pair (G, G_σ), we see that the first term vanishes because of $\lambda^k \neq \mu^k$, and that the third term vanishes because one of λ^k, μ^k must equal ω_4. (Recall that $L(\omega_4)$ is injective for G_σ: this was shown by Steinberg [21].)

Next we consider the case where the smallest integer k such that $\lambda^k \neq \mu^k$ is even, say $k = 2s$, but that there also exists $j > 2(s+1)$ with $\lambda^j \neq \mu^j$. Again, the 5-term Hochschild Serre sequences for the pairs $(\tilde{G}, \tilde{G}_\tau)$ and (G, G_σ) give us the reduction (if indeed $s \geq 1$)

$$\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu)) \cong \mathrm{Ext}^1_G(L(\sum_{i \geq s} 2^{i-s}[\sigma\lambda^{2i} + 2\lambda^{2i+1}]), L(\sum_{i \geq s} 2^{i-s}[\sigma\mu^{2i} + 2\mu^{2i+1}]))$$

Applying now the 5-term Hochschild-Serre sequence for the pair (G, G_σ), we see that the third term vanishes because of $\lambda^j \neq \mu^j$, and Lemma 4.4.2. Thus we obtain

$$\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu)) \cong \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\sum_{i \geq s} 2^{i-s}[\lambda^{2i} + \tau\lambda^{2i+1}]), \tilde{L}(\sum_{i \geq s} 2^{i-s}[\mu^{2i} + \tau\mu^{2i+1}]))$$

We then apply the 5-term Hochschild Serre sequence for the pair $(\tilde{G}, \tilde{G}_\tau)$. The first term vanishes since $\lambda^{2s} \neq \mu^{2s}$, and the third term vanishes because of $\lambda^j \neq \mu^j$, and Lemma 4.4.3. We turn now to the nontrivial cases.

a) After applying, if necessary, the reductions obtained from the 5-term Hochschild-Serre sequences for the pairs $(\tilde{G}, \tilde{G}_\tau)$ and (G, G_σ), we have

$$\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu)) \cong \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\sum_{i\geq s} 2^{i-s}[\sigma\lambda^{2i} + 2\lambda^{2i+1}]), \tilde{L}(\sum_{i\geq s} 2^{i-s}[\sigma\mu^{2i} + 2\mu^{2i+1}])).$$

Applying the 5-term Hochschild-Serre sequence for the pair (G, G_σ), we then obtain that the first and fourth terms vanish because of the hypothesis on the congruence classes, and thus

$$\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu))$$

$$\cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\sum_{i\geq s} 2^{i-s}[\lambda^{2i} + \tau\lambda^{2i+1}]), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\sum_{i\geq s} 2^{i-s}[\mu^{2i} + \tau\mu^{2i+1}])),$$

which vanishes unless $\lambda^i = \mu^i$ for all $i \geq 2s+1$, by Lemma 4.4.2. In the latter case it reduces to the hom group stated.

b) We reduce again as in the first step of part (a), if $s \geq 1$. We then apply the 5-term Hochschild-Serre sequence for the pair (G, G_σ). If $\lambda^{2s-1} = \mu^{2s-1} = \omega_4$, then the third term of the 5-term sequence vanishes because of the injectivity of $L(\omega_4)$ for G_σ. If λ^{2s} is in the same congruence class than μ^{2s}, then the third term vanishes because $\tilde{\omega}_1$ is in the nonzero congruence class. In any case, we obtain

$$\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\lambda), \tilde{L}(\mu)) \cong$$

$$\mathrm{Hom}_G(L(\sum_{i\geq s} 2^{i-s}[\lambda^{2i+1} + \sigma\lambda^{2(i+1)}]),$$

$$\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^{2s}), \tilde{L}(\mu^{2s}))^{(\tau^{-1})} \otimes L(\sum_{i\geq s} 2^{i-s}[\mu^{2i+1} + \sigma\mu^{2(i+1)}])),$$

which reduces to the stated result by Lemma 4.4.3.

Proposition 5 1 3 *Let G be the simply connected algebraic group of type B_4. Let λ, μ be a pair of dominant weights. Then*

$$Ext^1_{\tilde{G}}(L(\lambda), L(\mu)) \cong Ext^1_{\tilde{G}}(\tilde{L}(\tau\lambda), \tilde{L}(\tau\mu))$$

Proof This follows immediately from the 5-term Hochschild Serre sequence for the pair $(\tilde{G}, \tilde{G}_T)$: the fourth term vanishes since $\mathrm{Ext}^1_{\tilde{G}_T}(k, k) = 0$

The reader may now refer to the tables in section 5 4 to obtain all of the module extensions for B_4 and C_4

We now compute the module extensions for the Lie algebras of simply connected B_4 and C_4

Proposition 5 1 4 *Let $\tilde{G}$ be the simply connected algebraic group of type C_4. Let $\lambda = \lambda^0 + \tau\lambda^1$ and $\mu = \mu^0 + \tau\mu^1$ be 2-restricted weights. Then $Ext^1_{\tilde{G}_1}(\tilde{L}(\lambda), \tilde{L}(\mu))$ can be computed as follows*

a) If $\lambda^0 = \mu^0$, then the extension module is zero unless $\lambda^1 = \mu^1 = 0$, in which case

$$Ext^1_{\tilde{G}_1}(\tilde{L}(\lambda^0), \tilde{L}(\lambda^0)) \cong L(2\tilde{\omega}_1)$$

b) If $\lambda^0 \neq \mu^0$, then

$$\begin{aligned} &Ext^1_{\tilde{G}_1}(\tilde{L}(\lambda), \tilde{L}(\mu)) \\ &\cong Hom_{G_\sigma}(L(\lambda^1), Ext^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^0), \tilde{L}(\mu^0))^{(\tau^{-1})} \otimes L(\mu^1))^{(\tau)} \end{aligned}$$

Proof This is an immediate consequence of the 5-term sequence for the pair $(\tilde{G}_1, \tilde{G}_\tau)$, and the structure of the G_σ exts

Remark Because of the special nature of the $\tilde{G}_\tau$ exts, the hom modules in (b) can be computed easily by making the following observations

i) G_σ acts trivially on any Hom_{G_σ}, and thus any such hom-module must be the σ-twist of a $\tilde{G}$ module. However, there are no non-semisimple submodules of any of the $\mathrm{Ext}^1_{\tilde{G}_\tau}$ that are σ-twists.

ii) $L(\omega_4)$ is injective for G_σ, and thus $\mathrm{Ext}^1_G(L(\omega_4) \otimes L(\sigma\lambda), L(\omega_4) \otimes L(\sigma\mu)) \cong \mathrm{Ext}^1_{\tilde{G}}(L(\lambda), L(\mu))$ by the five-term sequence for (G, G_σ), for dominant weights λ, μ in $\tilde{X}(\tilde{T})$. In particular, note that we have $\mathrm{Ext}^1_{\tilde{G}}(L(\lambda), L(\mu)) = 0$ for all composition factors $L(\lambda), L(\mu)$ of the non-semisimple summands of the (untwisted) $\tilde{G}_\tau$ exts. However, taking G_σ -homs commutes with taking direct sums.

Proposition 5.1.5. *Let G be the simply connected algebraic group of type B_4. Let $\lambda = \lambda^0 + \sigma\lambda^1$ and $\mu = \mu^0 + \sigma\mu^1$ be 2-restricted weights. Then $Ext^1_{G_1}(L(\lambda), L(\mu))$ can be computed as follows:*

a) If $\lambda^0 \neq \mu^0$, then the extension module is zero.

b) If $\lambda^0 = \mu^0 = \omega_4$, then

$$Ext^1_{G_1}(L(\lambda), L(\mu))$$

$$\cong Ext^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\sigma)}$$

c) If $\lambda^0 = \mu^0 = 0$, then

i)

$$Ext^1_{G_1}(L(\lambda), L(\mu))$$

$$\cong Ext^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\sigma)},$$

if λ^1, μ^1 are in the same congruence class, and

ii)

$$Ext^1_{G_1}(L(\lambda), L(\mu))$$

$$\cong Hom_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^1))^{(\sigma)}$$

$$\cong Hom_{\tilde{G}}(\tilde{L}(\lambda^1), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^1))$$

if λ^1, μ^1 are in different congruence classes

PROOF. This is an immediate consequence of the 5-term sequence for the pair (G_1, G_σ), and Lemma 4.4.2. For part (c)(ii), to show that the fourth term of the 5-term sequence vanishes, we use that $H^*(D_1, M) \cong H^*(\hat{G}_T, M)$ as D- modules for any $\tilde{G}$-module M (cf. §4.3 and Jantzen [12, II 9.16b])

§5.2 Ext Computation Tables for A_4

Propositions 5.1.1 - 5.1.3 allow us now to read off all of the extensions for the algebraic groups from appropriate tables of tensor product socles. For $G = A_4$, we have

Table 5-1 $E = L(\lambda_1) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1 + \lambda_2 + \lambda_4))^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2 + \lambda_4))^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_2))^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2 + \lambda_3), L(\lambda_1 + \lambda_2))^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_4))^{(2^{-1})}$

Table 5-2 $E = L(\lambda_2) \cong \mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_4), k)^{(2^{-1})}$

Of course, no table is needed for $E = k \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_2 + \lambda_3))^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_3), k)^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4))^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_4), L(\lambda_1))^{(2^{-1})}$

All other $\mathrm{Ext}^1_{G_1}$ modules for A_4 are either zero or duals of those listed above

TABLE 5-1 $E = L(\lambda_1)$

product	socle
$E \otimes L(\lambda_1)$	$L(\lambda_2)$
$E \otimes L(\lambda_4)$	$k \oplus L(\lambda_1 + \lambda_4)$
$E \otimes L(\lambda_2)$	$L(\lambda_2) \oplus L(\lambda_1 + \lambda_2)$
$E \otimes L(\lambda_3)$	$L(\lambda_4)$
$E \otimes L(\lambda_1 + \lambda_4)$	$L(\lambda_1)$
$E \otimes L(\lambda_2 + \lambda_4)$	$L(\lambda_3 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3)$	$L(\lambda_2 + \lambda_3)$
$E \otimes L(\lambda_3 + \lambda_4)$	$L(\lambda_3)$
$E \otimes L(\lambda_1 + \lambda_2)$	$L(\lambda_1 + \lambda_3)$
$E \otimes L(\lambda_2 + \lambda_3)$	$L(\lambda_2 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_3)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_4)$	$L(\lambda_1 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3 + \lambda_4)$	$L(\lambda_1 + \lambda_3) \oplus L(\lambda_2 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_3)$	$L(\lambda_1 + \lambda_2 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_3 + \lambda_4)$	$L(\lambda_2 + \lambda_3) \oplus L(\rho)$
$E \otimes L(\rho)$	$L(\lambda_1 + \lambda_2 + \lambda_3)$

TABLE 5-2 $E = L(\lambda_2)$

product	socle
$E \otimes L(\lambda_1)$	$L(\lambda_3) \oplus L(\lambda_1 + \lambda_2)$
$E \otimes L(\lambda_4)$	$L(\lambda_1)$
$E \otimes L(\lambda_2)$	$L(\lambda_4) \oplus L(\lambda_1 + \lambda_3)$
$E \otimes L(\lambda_3)$	$k \oplus L(\lambda_1 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_4)$	$L(\lambda_2) \oplus L(\lambda_3 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_4)$	$L(\lambda_3) \oplus L(\lambda_1 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3)$	$L(\lambda_2 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_3)$
$E \otimes L(\lambda_3 + \lambda_4)$	$L(\lambda_4) \oplus L(\lambda_2 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_2)$	$L(\lambda_2 + \lambda_3) \oplus L(\lambda_1 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_3)$	$L(\lambda_3 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_4)$	$L(\lambda_1 + \lambda_3) \oplus L(\lambda_2 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3 + \lambda_4)$	$L(\lambda_2 + \lambda_3) \oplus L(\rho)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_3)$	$L(\lambda_1 + \lambda_2) \oplus L(\lambda_1 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_3 + \lambda_4)$	$L(\lambda_2 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_3)$
$E \otimes L(\rho)$	$L(\lambda_1 + \lambda_2 + \lambda_4)$

§5 3 Ext Computation Tables for D_4

We record the following tables for use in the computation of extensions for simply connected D_4 (via Proposition 5 1 1).

Table 5-3: $E = L(\delta_3) \oplus L(\delta_4) \cong \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2))^{(2^{-1})}$

Table 5-4. $E = k \oplus L(\delta_1) \cong \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3 + \delta_4))^{(2^{-1})}$

Table 5-5 $E = 2k \oplus L(\delta_1) \oplus L(\delta_3) \oplus L(\delta_4) \cong \mathrm{Ext}^1_{D_1}(k, L(\delta_2))^{(2^{-1})}$

Table 5-6. $E = k \oplus L(\delta_2) \cong \mathrm{Ext}^1_{D_1}(k, L(\delta_1 + \delta_3 + \delta_4))^{(2^{-1})}$

Of course, no table is needed for $E = k \cong \mathrm{Ext}^1_{D_1}(L(\delta_2), L(\delta_1 + \delta_3 + \delta_4))^{(2^{-1})}$
$\cong \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4))^{(2^{-1})}$

All other $\mathrm{Ext}^1_{D_1}$ modules for D_4 are either zero or duals or images under the graph automorphisms of those listed above

TABLE 5-3 $E = L(\delta_3) \oplus L(\delta_4)$

product	socle
$E \otimes L(\delta_1)$	$L(\delta_4) \oplus L(\delta_3)$
$E \otimes L(\delta_3)$	$k \oplus L(\delta_1)$
$E \otimes L(\delta_2)$	$L(\delta_1 + \delta_4) \oplus L(\delta_2 + \delta_3) \oplus L(\delta_1 + \delta_3) \oplus L(\delta_2 + \delta_4)$
$E \otimes L(\delta_3 + \delta_4)$	$L(\delta_2 + \delta_4) \oplus L(\delta_2 + \delta_3)$
$E \otimes L(\delta_1 + \delta_4)$	$L(\delta_2) \oplus L(\delta_1 + \delta_2)$
$E \otimes L(\delta_1 + \delta_2)$	$L(\delta_1 + \delta_3) \oplus L(\delta_2 + \delta_4) \oplus L(\delta_1 + \delta_4) \oplus L(\delta_2 + \delta_3)$
$E \otimes L(\delta_2 + \delta_3)$	$L(\delta_2) \oplus L(\delta_3 + \delta_4) \oplus L(\delta_1 + \delta_2)$
$E \otimes L(\delta_1 + \delta_3 + \delta_4)$	$L(\delta_1 + \delta_2 + \delta_4) \oplus L(\delta_1 + \delta_2 + \delta_3)$
$E \otimes L(\delta_2 + \delta_3 + \delta_4)$	$L(\delta_1 + \delta_2 + \delta_3) \oplus L(\delta_1 + \delta_2 + \delta_4)$
$E \otimes L(\delta_1 + \delta_2 + \delta_4)$	$L(\delta_1 + \delta_2 + \delta_4) \oplus L(\rho) \oplus L(\delta_2 + \delta_3 + \delta_4)$
$E \otimes L(\rho)$	$L(\delta_1 + \delta_2 + \delta_4) \oplus L(\delta_1 + \delta_2 + \delta_3)$

TABLE 5-4 $E = k \oplus L(\delta_1)$

product	socle
$E \otimes L(\delta_1)$	$L(\delta_1) \oplus k$
$E \otimes L(\delta_3)$	$L(\delta_3) \oplus L(\delta_4)$
$E \otimes L(\delta_2)$	$L(\delta_2) \oplus L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_3+\delta_4)$	$L(\delta_3+\delta_4) \oplus L(\delta_2)$
$E \otimes L(\delta_1+\delta_4)$	$L(\delta_1+\delta_4) \oplus L(\delta_2+\delta_4)$
$E \otimes L(\delta_1+\delta_2)$	$L(\delta_1+\delta_2) \oplus L(\delta_2)$
$E \otimes L(\delta_2+\delta_3)$	$L(\delta_2+\delta_3) \oplus L(\delta_1+\delta_3) \oplus L(\delta_2+\delta_4)$
$E \otimes L(\delta_1+\delta_3+\delta_4)$	$L(\delta_1+\delta_3+\delta_4) \oplus L(\delta_2+\delta_3+\delta_4)$
$E \otimes L(\delta_2+\delta_3+\delta_4)$	$L(\delta_2+\delta_3+\delta_4) \oplus L(\delta_1+\delta_2+\delta_4) \oplus L(\rho)$
$E \otimes L(\delta_1+\delta_2+\delta_4)$	$L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$
$E \otimes L(\rho)$	$L(\rho) \oplus L(\delta_2+\delta_3+\delta_4)$

TABLE 5-5 $E = 2k \oplus L(\delta_1) \oplus L(\delta_3) \oplus L(\delta_4)$

product	socle
$E \otimes L(\delta_1)$	$2L(\delta_1) \oplus k \oplus L(\delta_4) \oplus L(\delta_2)$
$E \otimes L(\delta_2)$	$2L(\delta_2) \oplus L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2) \oplus L(\delta_1+\delta_4) \oplus$ $L(\delta_2+\delta_3) \oplus L(\delta_1+\delta_3) \oplus L(\delta_2+\delta_4)$
$E \otimes L(\delta_3+\delta_4)$	$2L(\delta_3+\delta_4) \oplus L(\delta_2) \oplus$ $L(\delta_2+\delta_4) \oplus L(\delta_2+\delta_3)$
$E \otimes L(\delta_1+\delta_2)$	$2L(\delta_1+\delta_2) \oplus L(\delta_2) \oplus L(\delta_1+\delta_3) \oplus$ $L(\delta_2+\delta_4) \oplus L(\delta_1+\delta_4) \oplus L(\delta_2+\delta_3)$
$E \otimes L(\delta_1+\delta_3+\delta_4)$	$2L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_2+\delta_3+\delta_4) \oplus$ $L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$
$E \otimes L(\delta_2+\delta_3+\delta_4)$	$2L(\delta_2+\delta_3+\delta_4) \oplus L(\delta_1+\delta_3+\delta_4) \oplus$ $L(\rho) \oplus L(\delta_1+\delta_2+\delta_3) \oplus L(\delta_1+\delta_2+\delta_4)$
$E \otimes L(\rho)$	$2L(\rho) \oplus L(\delta_2+\delta_3+\delta_4) \oplus$ $L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$

TABLE 5-6 $E = k \oplus L(\delta_2)$

product	socle
$E \otimes L(\delta_1)$	$L(\delta_1) \oplus L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_2)$	$2L(\delta_2) \oplus k$
$E \otimes L(\delta_3+\delta_4)$	$L(\delta_3+\delta_4) \oplus L(\delta_1) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_1+\delta_2)$	$2L(\delta_1+\delta_2) \oplus L(\delta_1) \oplus L(\delta_2+\delta_4) \oplus L(\delta_2+\delta_3+\delta_4)$
$E \otimes L(\delta_1+\delta_3+\delta_4)$	$2L(\delta_1+\delta_3+\delta_4) \oplus L(\rho)$
$E \otimes L(\delta_2+\delta_3+\delta_4)$	$2L(\delta_2+\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\rho)$	$3L(\rho) \oplus L(\delta_1+\delta_3+\delta_4)$

§5.4 Ext Computation Tables for B_4 and C_4

The following tables allow us to use Propositions 5.1.2 and 5.1.3 to compute all of the extensions for simply connected B_4 and C_4

Table 5-7 $E = \mathrm{Ext}^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_1), L(\tilde{\omega}_1+\tilde{\omega}_2))^{(r^{-1})}$

Table 5-8 $E = \mathrm{Ext}^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_3))^{(r^{-1})}$

Table 5-9 $E = \mathrm{Ext}^1_{\tilde{G}_r}(k, \tilde{L}(\tilde{\omega}_2))^{(r^{-1})}$

Table 5-10 $E = \mathrm{Ext}^1_{\tilde{G}_r}(k, \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_3))^{(r^{-1})}$

Of course, no table is needed for $E = k \cong \mathrm{Ext}^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_2), \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_3))^{(r^{-1})}$ $\cong \mathrm{Ext}^1_{\tilde{G}_r}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))^{(r^{-1})}$

All other $\mathrm{Ext}^1_{\tilde{G}_r}$ modules for C_4 are either zero or duals of those listed above. To apply the result of Proposition 5.1.2, we will also need the $\tilde{G}$ socles of tensor products of

$E = \tilde{L}(\tilde{\omega}_1) \cong \mathrm{Ext}^1_{G_\sigma}(k,k)^{(\sigma^{-1})}$ with τ-restricted $\tilde{G}$-modules (Table 5-11). The tables in this section are computed using the results of Chapter 3, the fact that

$$\mathrm{Hom}_{D_1}(k, M^{(\sigma)}) \cong \mathrm{Hom}_{G_1}(k, M^{(\sigma)})$$

as D-modules for (for arbitrary $\tilde{G}$-modules M) together with the following lemma. We will also use the fact that $L(\omega_4)$ is injective for G_σ.

Lemma 5.4.1. *Let M be one of the non-semisimple $\mathrm{Ext}^1_{G_\tau}$ modules listed in Lemma 5.3.1. Let μ, ν be 2-restricted weights for $G = B_4$ which are σ-twisted (i.e., in the span of the fundamental dominant weights corresponding to the long simple roots). Then*

$$\mathrm{Hom}_G(L(\nu), M \otimes L(\mu)) \cong \mathrm{Hom}_G(L(\nu), \mathrm{soc}_G(M) \otimes L(\mu))$$

Proof. We have

$$\mathrm{Hom}_G(L(\nu), M \otimes L(\mu)) \cong \mathrm{Hom}_G(L(\nu) \otimes L(\mu)^*, M)$$

However, $L(\nu) \otimes L(\mu)^*$ is an σ twist of some $\tilde{G}$-module, whereas M has no non semisimple submodules which are σ twists.

TABLE 5-7 $E = \mathrm{Ext}^1_{G_r}(\bar{L}(\bar{\omega}_1), \bar{L}(\bar{\omega}_1 + \bar{\omega}_2))^{(r^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	$L(\omega_1 + \omega_4)$
$E \otimes L(\omega_2)$	$L(\omega_2 + \omega_4)$
$E \otimes L(\omega_3)$	$L(\omega_3 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2)$	$L(\omega_1 + \omega_2 + \omega_4)$
$E \otimes L(\omega_1 + \omega_3)$	$L(\omega_1 + \omega_2 + \omega_4)$
$E \otimes L(\omega_2 + \omega_3)$	$L(\omega_2 + \omega_3 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2 + \omega_3)$	$L(\rho)$
$E \otimes L(\omega_4)$	k
$E \otimes L(\omega_1 + \omega_4)$	$L(\omega_1)$
$E \otimes L(\omega_2 + \omega_4)$	$L(\omega_2)$
$E \otimes L(\omega_3 + \omega_4)$	$L(\omega_3)$
$E \otimes L(\omega_1 + \omega_2 + \omega_4)$	$L(\omega_1 + \omega_2)$
$E \otimes L(\omega_1 + \omega_3 + \omega_4)$	$L(\omega_1 + \omega_3)$
$E \otimes L(\omega_2 + \omega_3 + \omega_4)$	$L(\omega_2 + \omega_3)$
$E \otimes L(\rho)$	$L(\omega_1 + \omega_2 + \omega_3)$

TABLE 5.8 $E = \mathrm{EXT}^1_{G_r}(\bar{L}(\tilde{\omega}_1), \bar{L}(\tilde{\omega}_3))^{(\tau^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	k
$E \otimes L(\omega_2)$	$L(\omega_1+\omega_2) \oplus L(\omega_3)$
$E \otimes L(\omega_3)$	$L(\omega_2)$
$E \otimes L(\omega_1+\omega_2)$	$L(\omega_2)$
$E \otimes L(\omega_1+\omega_3)$	$L(\omega_2+\omega_3)$
$E \otimes L(\omega_2+\omega_3)$	$L(\omega_1+\omega_3) \oplus L(\omega_1+\omega_2+\omega_3)$
$E \otimes L(\omega_1+\omega_2+\omega_3)$	$L(\omega_2+\omega_3)$
$E \otimes L(\omega_4)$	$L(\omega_4) \oplus L(\omega_1+\omega_4)$
$E \otimes L(\omega_1+\omega_4)$	$L(\omega_1+\omega_4) \oplus L(\omega_4)$
$E \otimes L(\omega_2+\omega_4)$	$L(\omega_2+\omega_4) \oplus L(\omega_1+\omega_2+\omega_4) \oplus L(\omega_2+\omega_4)$
$E \otimes L(\omega_3+\omega_4)$	$L(\omega_3+\omega_4) \oplus L(\omega_2+\omega_4)$
$E \otimes L(\omega_1+\omega_2+\omega_4)$	$L(\omega_1+\omega_2+\omega_4) \oplus L(\omega_2+\omega_4)$
$E \otimes L(\omega_1+\omega_3+\omega_4)$	$L(\omega_1+\omega_3+\omega_4) \oplus L(\omega_2+\omega_3+\omega_4)$
$E \otimes L(\omega_2+\omega_3+\omega_4)$	$L(\omega_2+\omega_3+\omega_4) \oplus L(\omega_1+\omega_3+\omega_4) \oplus L(\rho)$
$E \otimes L(\rho)$	$L(\rho) \oplus L(\omega_2+\omega_3+\omega_4)$

TABLE 5-9 $E = \mathrm{Ext}^1_{G_r}(L(k, \bar{L}(\bar{\omega}_2))^{(r^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	$L(\omega_1) \oplus L(\omega_1+\omega_4)$
$E \otimes L(\omega_2)$	$L(\omega_2) \oplus L(\omega_2+\omega_4)$
$E \otimes L(\omega_3)$	$L(\omega_3) \oplus L(\omega_3+\omega_4)$
$E \otimes L(\omega_1+\omega_2)$	$L(\omega_1+\omega_2) \oplus L(\omega_1+\omega_2+\omega_4)$
$E \otimes L(\omega_1+\omega_3)$	$L(\omega_1+\omega_3) \oplus L(\omega_1+\omega_3+\omega_4)$
$E \otimes L(\omega_2+\omega_3)$	$L(\omega_2+\omega_3) \oplus L(\omega_2+\omega_3+\omega_4)$
$E \otimes L(\omega_1+\omega_2+\omega_3)$	$L(\omega_1+\omega_2+\omega_3) \oplus L(\rho)$
$E \otimes L(\omega_4)$	$k \oplus 2L(\omega_4) \oplus L(\omega_1+\omega_4)$
$E \otimes L(\omega_1+\omega_4)$	$L(\omega_1) \oplus 2L(\omega_1+\omega_4) \oplus L(\omega_4)$
$E \otimes L(\omega_2+\omega_4)$	$L(\omega_2) \oplus 2L(\omega_2+\omega_4) \oplus$ $L(\omega_1+\omega_2+\omega_4) \oplus L(\omega_3+\omega_4)$
$E \otimes L(\omega_3+\omega_4)$	$L(\omega_2) \oplus 2L(\omega_3+\omega_4) \oplus L(\omega_2+\omega_4)$
$E \otimes L(\omega_1+\omega_2+\omega_4)$	$L(\omega_1+\omega_2) \oplus 2L(\omega_1+\omega_2+\omega_4) \oplus L(\omega_2+\omega_4)$
$E \otimes L(\omega_1+\omega_3+\omega_4)$	$L(\omega_1+\omega_3) \oplus 2L(\omega_1+\omega_3+\omega_4) \oplus L(\omega_2+\omega_3+\omega_4)$
$E \otimes L(\omega_2+\omega_3+\omega_4)$	$L(\omega_2+\omega_3) \oplus 2L(\omega_2+\omega_3+\omega_4) \oplus$ $L(\omega_1+\omega_3+\omega_4) \oplus L(\rho)$
$E \otimes L(\rho)$	$L(\omega_1+\omega_2+\omega_3) \oplus 2L(\rho) \oplus L(\omega_2+\omega_3+\omega_4)$

TABLE 5-10 $E = \mathrm{Ext}^1_{G_r}(L(k, \bar{L}(\bar{\omega}_1+\bar{\omega}_3))^{(r^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	$L(\omega_1) \oplus L(\omega_3) \oplus L(\omega_1+\omega_2)$
$E \otimes L(\omega_2)$	$2L(\omega_2) \oplus k$
$E \otimes L(\omega_3)$	$L(\omega_3) \oplus L(\omega_1) \oplus L(\omega_1+\omega_2)$
$E \otimes L(\omega_1+\omega_2)$	$2L(\omega_1+\omega_2) \oplus L(\omega_1) \oplus L(\omega_2) \oplus L(\omega_2+\omega_3)$

TABLE 5-10 (CONTINUED)

product	G-socle
$E \otimes L(\omega_1+\omega_3)$	$2L(\omega_1+\omega_3) \oplus L(\omega_1+\omega_2+\omega_3)$
$E \otimes L(\omega_2+\omega_3)$	$2L(\omega_2+\omega_3) \oplus L(\omega_1+\omega_2)$
$E \otimes L(\omega_1+\omega_2+\omega_3)$	$3L(\omega_1+\omega_2+\omega_3) \oplus L(\omega_1+\omega_3)$
$E \otimes L(\omega_4)$	$L(\omega_4) \oplus L(\omega_3+\omega_4)$
$E \otimes L(\omega_1+\omega_4)$	$L(\omega_1+\omega_4) \oplus L(\omega_3+\omega_4) \oplus L(\omega_1+\omega_3+\omega_4)$
$E \otimes L(\omega_3+\omega_4)$	$2L(\omega_3+\omega_4) \oplus L(\omega_4)$
$E \otimes L(\omega_2+\omega_4)$	$L(\omega_2+\omega_4) \oplus L(\omega_1+\omega_4) \oplus L(\omega_1+\omega_3+\omega_4)$
$E \otimes L(\omega_1+\omega_2+\omega_4)$	$2L(\omega_1+\omega_2+\omega_4) \oplus L(\omega_1+\omega_4) \oplus$ $L(\omega_3+\omega_4) \oplus L(\omega_2+\omega_3+\omega_4)$
$E \otimes L(\omega_1+\omega_3+\omega_4)$	$2L(\omega_1+\omega_3+\omega_4) \oplus L(\rho)$
$E \otimes L(\omega_2+\omega_3+\omega_4)$	$2L(\omega_2+\omega_3+\omega_4) \oplus L(\omega_1+\omega_3+\omega_4)$
$E \otimes L(\rho)$	$3L(\rho) \oplus L(\omega_1+\omega_3+\omega_4)$

TABLE 5-11 $E = \mathrm{Ext}^1_{G_\sigma}(k,k)^{(\sigma^{-1})} \cong \tilde{L}(\tilde{\omega}_1)$

product	$\tilde{G}$-socle
$E \otimes \tilde{L}(\tilde{\omega}_1)$	k
$E \otimes \tilde{L}(\tilde{\omega}_2)$	$\tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2) \oplus \tilde{L}(\tilde{\omega}_3)$
$E \otimes \tilde{L}(\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_2)$
$E \otimes \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2)$	$\tilde{L}(\tilde{\omega}_2)$
$E \otimes \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$
$E \otimes \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_1+\tilde{\omega}_3) \oplus \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2+\tilde{\omega}_3)$
$E \otimes \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2+\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$

CHAPTER 6

1-COHOMOLOGY OF THE FINITE GROUPS

§6.1 p^n-Restricted "Mass" of a Module

We now consider the problem of computing 1-cohomology for the finite groups of rank 4. Most of our results will hinge on whether or not particular simple modules appear as composition factors of certain tensor products of simple modules, the main tool for this type of analysis will be the concept of module "mass", as first introduced in the papers of Sin [15, 16, 17, 18].

We must first define "mass" for modules over the algebraic group. In the following lemmas, we let G be an arbitrary semisimple, simply connected, algebraic group over an algebraically closed field of characteristic p. Let $\mathbf{E} = \mathbf{E}_{\mathbb{R}} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Fix some $f \in \mathbf{E}^*$ such that $f(\alpha_i) > 0$ for all $\alpha_i \in \Delta$ (e.g., we may take $f = (t\check{\rho}, \ )$, where $\rho = 1/2 \sum_{\alpha \in \Phi^+} \alpha_i$ and where t is the torsion coefficient of $X(T)/\mathbb{Z}\Phi$, this will ensure that m will take values in $\mathbb{Z}^+$). Define the (p-restricted) "mass" of a module, $m(V) \in \mathbb{R}$, for G-modules V as follows:

i) For $\lambda = \sum_{i=0}^{r} p^i \lambda_i \in X(T)^+$ (where $\lambda_i \in X_1(T)$ for all i), we let $m(\lambda) = \sum_{i=0}^{r} f(\lambda_i)$

ii) Define $m(V) = \sup\{m(\lambda) : L(\lambda)$ is a composition factor of $V\}$. (In particular, we have $m(L(\lambda)) = m(\lambda)$.)

In the notation established above, if we define the mass function by taking $f = (\check{\rho}, \ )$,

we have the masses for the simple restricted modules for A_4, B_4 and D_4 as listed in Tables 1-1 to 1-3.

Lemma 6.1.1. *Let $\lambda, \lambda' \in X(T)^+$, with $\lambda = \sum_{i=0}^{r} p^i \lambda_i$, $\lambda' = \sum_{i=0}^{r} p^i \lambda_i'$ (where $\lambda_i, \lambda_i' \in X_1(T)$ for all i). Then $m(L(\lambda) \otimes L(\lambda')) \leq m(\lambda) + m(\lambda')$ with equality if and only if $\lambda_i + \lambda_i' \in X_1(T)$ for all i, in which case $L(\lambda + \lambda')$ is the unique composition factor of $L(\lambda) \otimes L(\lambda')$ of greatest mass.*

Proof. Case 1: λ, λ' both p-restricted.

Suppose $V = L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\lambda')$. Then $\nu \preceq \lambda + \lambda'$ in the $\mathbf{Z}^+\Delta$ (usual) partial order. If $\nu = \sum p^i \nu_i$ ($\nu_i \in X_1(T)$), then we have $m(\nu) = \sum f(\nu_i) \leq \sum p^i f(\nu_i) = f(\nu) \leq f(\lambda + \lambda') = f(\lambda) + f(\lambda') = m(\lambda) + m(\lambda')$ with equality if and only if $\nu = \nu_0 \in X_1(T)$ and $\nu = \lambda + \lambda'$. Thus $m(L(\lambda) \otimes L(\lambda')) \leq m(\lambda) + m(\lambda')$ with equality if and only if $\nu = \lambda + \lambda' \in X_1(T)$.

Case 2: $\{\lambda, \lambda'\} \not\subseteq X_1(T)$.

We induct on the quantity $m(\lambda) + m(\lambda')$. Write $\lambda = \lambda_0 + p\bar{\lambda}$, $\lambda' = \lambda_0' + p\bar{\lambda}'$. Since mass is preserved under untwisting, we may assume that $\lambda_0 + \lambda_0' \neq 0$. Also, we have $\bar{\lambda} + \bar{\lambda}' \neq 0$ by assumption. Now,

$$m(L(\lambda) \otimes L(\lambda')) = m(L(\lambda_0) \otimes L(\lambda_0') \otimes L(p\bar{\lambda}) \otimes L(p\bar{\lambda}')) = m(L(\nu) \otimes L(\nu'))$$

for some composition factors $L(\nu)$, $L(\nu')$ of $L(\lambda_0) \otimes L(\lambda_0')$, $L(p\bar{\lambda}) \otimes L(p\bar{\lambda}')$ respectively. By induction then, $m(\nu) \leq m(\lambda_0) + m(\lambda_0')$ and $m(\nu') \leq m(p\bar{\lambda}) + m(p\bar{\lambda}')$. If equality holds in both, we would have that $\lambda_i + \lambda_i' \in X_1(T)$ for all i, that $L(\nu) = L(\lambda_0 + \lambda_0')$ and $L(\nu') = L(p\bar{\lambda} + p\bar{\lambda}')$ are the unique composition factors of greatest mass of $L(\lambda_0) \otimes L(\lambda_0')$ and $L(p\bar{\lambda}) \otimes L(p\bar{\lambda}')$, respectively, and thus that $L(\lambda + \lambda') = L(\nu) \otimes L(\nu')$ is the unique composition factor of $L(\lambda) \otimes L(\lambda')$ of greatest mass $m(\lambda + \lambda') = m(\lambda) + m(\lambda')$. Otherwise, $m(\nu) + m(\nu') < m(\lambda) + m(\lambda')$, so that the induction hypothesis could

be applied to $L(\nu) \otimes L(\nu')$ conclude that $m(L(\lambda) \otimes L(\lambda')) = m(L(\nu) \otimes L(\nu')) \leq m(\nu) + m(\nu') < m(\lambda) + m(\lambda')$ ∎

In the following corollary, define $\theta = \{\min_{\beta \in X_1(T) \setminus \{0\}} (m(\beta))\}$. This quantity will be used frequently throughout our mass arguments. Observe that $\theta = 2$ if $G = A_4$, $\theta = 3$ if $G = D_4$, and $\theta = 4$ if $G = B_4$.

COROLLARY 6.1.2. *If $\lambda, \lambda' \in X_1(T)$ and $L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\lambda')$ with $\nu \notin X_1(T)$, then $m(L(\nu)) \leq m(\lambda) + m(\lambda') - (p-1) \cdot \theta$*

PROOF. Suppose $\nu = \sum_{i=0}^{r} p^i \nu_i$ is the p-adic expansion of ν. We rewrite the inequality from the proof of Case 1: $m(\lambda) + m(\lambda') - m(\nu) = f(\lambda + \lambda') - \sum f(\nu_i) \geq f(\nu) - \sum f(\nu_i) = \sum p^i f(\nu_i) - \sum f(\nu_i) = \sum (p^i - 1) m(\nu_i) \geq (p-1) m(\nu_k)$ for some $1 \leq k \leq r$ and $\nu_k \neq 0$, by assumption on ν ∎

We may also define, for any $k \in \mathbb{N}$, the p^k restricted mass, by letting $m_{p^k}(\lambda) = \sum_{i=0}^{r} p^{r(i,k)} f(\lambda_i)$, (and extending to nonsimple modules as in (ii) above,) where $r(i,k)$ is the least non-negative residue of i mod k. It is an easy exercise to check that statements analogous to Lemma 6.1.1 and Corollary 6.1.2 hold for p^k restricted mass. There is a natural way of extending the definition of mass to $G(n)$-modules by representing the simple modules as restrictions to $G(n)$ of G-modules with 2^n -restricted highest weight, it can then be shown that the p^k restricted mass of a G module is $\geq$ to the p^k restricted mass (as $G(n)$ -module) of its restriction to $G(n)$

LEMMA 6.1.3. *Let $\lambda = \sum_{i=0}^{n-1} p^i \lambda_i = \lambda^0 + p\bar{\lambda}$, $\mu = \sum_{i=0}^{n-1} p^i \mu_i = \mu^0 + p\bar{\mu}$, $\nu = \sum_{i=1}^{n-1} p^i \nu_i = p\bar{\nu}$, where $\lambda_0 \neq \mu_0$, and $m(\lambda_i) \geq m(\mu_i)$ for all $i = 1, \ldots, n-1$. If $L(\lambda)$ is a composition factor of*

$$L(\nu) \otimes L(\mu) = L(p\bar{\nu}) \otimes L(\mu),$$

as $G(n)$-modules, then $(p^n - 1) \cdot \theta \leq (m(\mu_0) - m(\lambda_0)) + \sum_{i=1}^{n-1} p^i m(\nu_i)$

Proof Since $\lambda_0 \neq \mu_0$, $L(\lambda)$ cannot be a composition factor (as G-module) of

$$L(\nu) \otimes L(\mu) \quad \cong L(\mu_0) \otimes [L(p\bar{\nu}) \otimes L(p\bar{\mu})],$$

by Steinberg's tensor product theorem. Therefore, we must have that $L(\lambda)$ is a composition factor of $\mathrm{res}_{G(n)}(L(\omega))$, for some non p^n restricted weight ω such that $L(\omega)$ is a G-composition factor of $L(\nu) \otimes L(\mu)$. This implies that

$$m_{p^n}(L(\lambda)) = \sum_{i=0}^{n-1} p^i m(\lambda_i) \leq m_{p^n}(L(\omega))$$

$$\leq m_{p^n}(L(\nu)) + m_{p^n}(L(\mu)) - (p^n - 1) \cdot \theta$$

$$= \sum_{i=1}^{n-1} p^i m(\nu_i) + \sum_{i=0}^{n-1} p^i m(\mu_i) - (p^n - 1) \cdot \theta$$

$$\leq \sum_{i=1}^{n-1} p^i m(\nu_i) + (m(\mu_0) - m(\lambda_0)) + \sum_{i=0}^{n-1} p^i m(\lambda_i) - (p^n - 1) \cdot \theta$$

whence

$$(p^n - 1) \cdot \theta \leq (m(\mu_0) - m(\lambda_0)) + \sum_{i=1}^{n-1} p^i m(\nu_i)$$

∎

§6.2 Projective Covers

Lemma 6.2.1

A) (D_4 version) Let $I_1, I_2, \ldots, I_{14}$, and R be disjoint subsets of $N=\{0,1,\ldots,n-1\}$, and let $i \in N$. Then $A_i \otimes (\Theta^a_{I_1} \otimes \Theta^b_{I_2} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$ contains no composition factor of the form $\Theta^a_{I'_1} \otimes \Theta^b_{I'_2} \otimes \cdots \otimes \Gamma^c_{I'_{14}} \otimes S_{R'}$ with $|R'| > |R| + 1$, if A denotes any of $\Theta^{abc}, \mu, \Lambda^{abc}, \Delta^{abc}, \Psi$, and no composition factor of the above form with $|R'| > |R|+2$,

if A denotes any of Γ^{abc}, S. Furthermore, if $i \in N \setminus (I_1 \cup \cdots \cup I_{14} \cup R)$, then $(S_i \otimes S_i) \otimes (\Theta^a_{I_1} \otimes \Theta^b_{I_2} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$ contains no composition factor of the form S_T with $|T| > |R| + 2$.

B) (A_4 version) Let $I_1, I_2, \ldots, I_{14}$, and R be disjoint subsets of $N = \{0, 1, \ldots, n-1\}$, and let $i \in N$. Then $A_i \otimes (\Theta_{I_1} \otimes \Theta^*_{I_2} \otimes \cdots \otimes \Gamma^*_{I_{14}} \otimes S_R)$ contains no composition factor of the form $\Theta_{I_1'} \otimes \Theta^*_{I_2'} \otimes \cdots \otimes \Gamma^*_{I_{14}'} \otimes S_{R'}$ with $|R'| > |R| + 1$, if A denotes any of $\Theta, \Theta^*, \Lambda, \Lambda^*\mu, \Delta, \Delta^*, \Sigma, \Sigma^*, \Psi, \Upsilon, \Upsilon^*$, and no composition factor of the above form with $|R'| > |R| + 2$, if A denotes any of Γ, Γ^*, S. Furthermore, if $i \in N \setminus (I_1 \cup \cdots \cup I_{14} \cup R)$, then $(S_i \otimes S_i) \otimes (\Theta_{I_1} \otimes \Theta^*_{I_2} \otimes \cdots \otimes \Gamma^*_{I_{14}} \otimes S_R)$ contains no composition factor of the form S_T with $|T| > |R| + 2$.

Proof. We induct on the quantity $m(A_i) + m(\Theta^a_{I_1} \otimes \Theta^b_{I_2} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$. If $i \notin I_1 \cup \cdots \cup I_{14} \cup R$, there is nothing to prove, so we analyze the filtration of $A_i \otimes (\Theta^a_{I_1} \otimes \Theta^b_{I_2} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R) = (A_i \otimes \Upsilon_i) \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Upsilon_{U \setminus \{i\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$, resulting from a composition series of $A_i \otimes \Upsilon_i$, for each choice of Υ corresponding to the various possibilities $i \in I_1, i \in I_2$, etc., and the various possible choices for A. Inspection of Table 2-1 shows that the resulting filtration factors are of one of the following 7 forms

i) Irreducible of the form $\Theta^a_{I_1'} \otimes \Theta^b_{I_2'} \otimes \cdots \otimes \Gamma^c_{I_{14}'} \otimes S_{R'}$ with $|R'| \leq |R| + 1$.

ii) Of the form $A'_j \otimes (\Theta^a_{I_1'} \otimes \cdots \otimes S_{R'})$ with $R' \subseteq R$, and $A' \neq \Gamma^{abc}, S$.

iii) Of the form $A'_j \otimes (\Theta^a_{I_1'} \otimes \cdots \otimes S_{R'})$ with $|R'| \leq |R| - 1$, and $A' = \Gamma^{abc}, S$.

iv) Of the form $A'_j \otimes (\Theta^a_{I_1'} \otimes \cdots \otimes S_{R'})$ with $R' = R$, and $A' = \Gamma^{abc}$. These only occur if $A = \Gamma^{abc}, S$.

v) Of the form $A'_j \otimes (\Theta^a_{I_1'} \otimes \cdots \otimes S_{R'})$ with $|R'| = |R| + 1$, and $A' \neq \Gamma^{abc}, S$. These occur only if $A = \Gamma^{abc}, S$.

vi) Of the form $A'_{i+1} \otimes A''_{i+2} \otimes (\Theta^a_{I'_1} \otimes \cdots \otimes S_{R'})$ with $|R'| = |R| - 1$, and $A', A'' \neq \Gamma^{abc}, S$

vii) Of the form $A'_{i+1} \otimes A''_{i+2} \otimes (\Theta^a_{I'_1} \otimes \cdots \otimes S_{R'})$ with $R' = R$, and $A', A'' \neq \Gamma^{abc}, S$. These occur only if $A = \Gamma^{abc}, S$.

In cases (i)-(v), we have

$$m(A'_j) + m(\Theta^a_{I'_1} \otimes \cdots \otimes S_{R'}) < m(A_i) + m(\Theta^a_{I_1} \otimes \cdots \otimes S_R),$$

and in cases (vi) and (vii), we have

$$m(A'_{i+1}) + m(A''_{i+2}) + m(\Theta^a_{I'_1} \otimes \cdots \otimes S_{R'}) < m(A_i) + m(\Theta^a_{I_1} \otimes \cdots \otimes S_R),$$

by Corollary 6.1.2.

Thus, we may apply the induction hypothesis (twice if necessary.)

Finally, to prove the last assertion, we examine the composition factors of $S_i \otimes S_i$ (cf. Table 2-1). We observe that we may apply the first assertion of the theorem at most twice in succession to terms of the form $A_j \otimes (\Theta^a_{I_1} \otimes \Theta^b_{I_2} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$ to obtain the result. The proof for $G = A_4$ is similar, with Γ, Γ^* playing the roles of Γ^{abc}. ∎

We will need information about the structure of the module $A_i \otimes S_R$. A restriction on which composition factors can appear in the head and socle is obtained by determining the decomposition into (projective) indecomposables of $A_i \otimes S_N$. In the following, $P(M)$ denotes the projective cover of M.

Lemma 6.2.2A (D_4 version)

a) $\Theta^a_i \otimes S_N \cong P(\Gamma^a_i \otimes S_{N\setminus\{i\}})$

b) $\Gamma^a_i \otimes S_N \cong P(\Theta^a_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Delta^a_i \otimes S_{N\setminus\{i\}}) \oplus 4P(\Delta^a_i \otimes S_{N\setminus\{i\}}) \oplus 4P(\Gamma^a_i \otimes$

$S_{N\setminus\{i\}}) \oplus 2P(\Gamma_i^a\Gamma_{i+1}^b \otimes S_{N\setminus\{i,i+1\}}) \oplus 2P(\Gamma_i^a\Gamma_{i+1}^c \otimes S_{N\setminus\{i,i+1\}})$

c) $\mu_i \otimes S_N \cong P(\Psi_i \otimes S_{N\setminus\{i\}}) \oplus 2S_N$

d) $\Psi_i \otimes S_N \cong P(\mu_i \otimes S_{N\setminus\{i\}}) \oplus 6S_N \oplus 2P(\Gamma_{i+1}^{abc} \otimes S_{N\setminus\{i+1\}})$

e) $\Lambda_i^a \otimes S_N \cong P(\Delta_i^a \otimes S_{N\setminus\{i\}}) \oplus 2P(\Gamma_i^a \otimes S_{N\setminus\{i\}})$

f) $\Delta_i^a \otimes S_N \cong P(\Lambda_i^a \otimes S_{N\setminus\{i\}}) \oplus 2P(\Delta_i^a \otimes S_{N\setminus\{i\}}) \oplus 4P(\Gamma_i^a \otimes S_{N\setminus\{i\}})$

PROOF

Let A denote one of the restricted module types $\Theta^a, \ldots, \Gamma^c$. We have

$$\dim_k(\mathrm{Hom}_{kG(n)}(A_i \otimes S_N, \Theta_{I_1}^a \otimes \Theta_{I_2}^b \otimes \cdots \otimes \Gamma_{I_{14}}^c \otimes S_R)$$

$$= \dim_k(\mathrm{Hom}_{kG(n)}(S_N, A_i^* \otimes (\Theta_{I_1}^a \otimes \Theta_{I_2}^b \otimes \cdots \otimes \Gamma_{I_{14}}^c \otimes S_R))$$

$=$ multiplicity of S_N as a composition factor of $A_i^* \otimes (\Theta_{I_1}^a \otimes \Theta_{I_2}^b \otimes \cdots \otimes \Gamma_{I_{14}}^c \otimes S_R)$ since S_N is simple and projective. The result can be nonzero only if $i \in I_1 \cup \ldots I_{14} \cup R$ unless $A = S$ (and $R = N \setminus \{i\}$.) We therefore wish to consider the filtration factors of $(A_i^* \otimes B_i) \otimes (\Theta_{I_1}^a \otimes \cdots \otimes B_{U\setminus\{i\}} \otimes \cdots \otimes \Gamma_{I_{14}}^c \otimes S_R)$ resulting from composition factors of $A_i^* \otimes B_i$ for various choice of restricted module type B, corresponding to the various possibilities $i \in I_1, \ldots, i \in I_{14}, i \in R$. The composition factors of $A_i^* \otimes B_i$ are of the form $\Pi_i \Omega_{i+1} \Omega'_{i+2}$. By Lemma 6.1.3, we see that S_N cannot be a composition factor of the resulting filtration factor unless $\Pi = S$, for otherwise we would have

$$(2^n - 1) \cdot 8 \leq (m(\Pi) - m(S)) + 2m(\Omega) + 4m(\Omega')$$

$$= m_{2^e}(\Pi\Omega_1\Omega'_2) - m(S) \leq m(A) + m(B) - m(S) \leq 14,$$

whereas we are assuming $n > 2$

a) $A = \Theta^a$

The result is nonzero only if $I_{12} = \{i\}$ (and $R = N \setminus \{i\}$), and multiplicity of S_N in $(\Theta_i^a \otimes \Gamma_i^a) \otimes S_{N\setminus\{i\}}$ is equal to one

b) $A = \Gamma^a$

The result can be nonzero only if $I_1 = \{i\}$ (and $R = N \setminus \{i\}$), $I_5 = \{i\}$ (and $R = N \setminus \{i\}$), $I_8 = \{i\}$ (and $R = N \setminus \{i\}$), or $I_{12} = \{i\}$ If $I_{12} = i$, we have that S_N can occur as a composition factor of $(\Gamma_i^a \otimes \Gamma_i^a) \otimes (\Theta_{I_1}^a \otimes \ldots \otimes \Gamma_{I_{12}\setminus\{i\}}^a \otimes \ldots \otimes \Gamma_{I_{14}}^c \otimes S_R)$ only in those filtration factors resulting from composition factors of $\Gamma_i^a \otimes \Gamma_i^a$ that are isomorphic to $S_i, S_i\Theta_{i+1}^b$, or $S_i\Theta_{i+1}^c$ For the latter two types of composition factors, the argument involving Lemma 6 1 3 shows that S_N can be a composition factor of the resulting filtration factor only if $I_2 = \{i+1\}$, and $I_3 = \{i+1\}$ (and $R = N \setminus \{i, i+1\}$)) respectively

c-g) The arguments are the same as those in a),b)

h) The above argument involving Lemma 6 1 3 is applied to show that a nonzero result is obtained only if $i \in I_{12}$, and then applied a second time to the filtration factors $\Theta_{i+1}^a \otimes (\Theta_{I_1}^a \otimes \ldots \otimes \Gamma_{I_{12}\setminus\{i\}}^a \otimes \ldots \otimes \Gamma_{I_{14}}^c \otimes S_{R\cup\{i\}})$ to show that a nonzero result is obtained only if $i+1 \in I_{12}$ (and $R = N \setminus \{i, i+1\}$.)

i) Argue as in b) ∎

Lemma 6 2 2B (A_4 **version**)

a) $\Theta_i \otimes S_N \cong P(\Gamma_i \otimes S_{N\setminus\{i\}})$

b) $\Gamma_i \otimes S_N \cong P(\Theta_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Delta_i \otimes S_{N\setminus\{i\}}) \oplus 6P(\Gamma_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Gamma_i^a\Gamma_{i+1} \otimes$

$S_{N\setminus\{i,i+1\}})$

c) $\mu_i \otimes S_N \cong P(\Psi_i \otimes S_{N\setminus\{i\}}) \oplus 4S_N$

d) $\Psi_i \otimes S_N \cong P(\mu_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Psi_i \otimes S_{N\setminus\{i\}}) \oplus 4S_N$

e) $\Lambda_i \otimes S_N \cong P(\Upsilon_i \otimes S_{N\setminus\{i\}})$

f) $\Upsilon_i \otimes S_N \cong P(\Lambda_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Sigma_i \otimes S_{N\setminus\{i\}}) \oplus 3P(\Upsilon_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Gamma_{i+1} \otimes S_{N\setminus\{i+1\}})$

g) $\Delta_i \otimes S_N \cong P(\Delta_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Gamma_i \otimes S_{N\setminus\{i\}})$

h) $\Sigma_i \otimes S_N \cong P(\Sigma_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Upsilon_i \otimes S_{N\setminus\{i\}})$

Proof

The proof is the same as that of Lemma 6.2.2A. ∎

Lemma 6.2.2C (B_4 version)

a) $\Theta_i \otimes S_N \cong P((\Gamma\sigma)_i \otimes S_{N\setminus\{i\}})$

b) $\Gamma_i \otimes S_N \cong P((\Theta\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\Lambda\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 4P((\Delta\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 4P((\Gamma\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\Gamma\sigma)_i\Sigma_{i+1} \otimes S_{N\setminus\{i,i+1\}})$

c) $\mu_i \otimes S_N \cong P((\Psi\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2S_N$

d) $\Psi_i \otimes S_N \cong P((\mu\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 6S_N \oplus 2P((\Gamma\sigma)_{i+1} \otimes S_{N\setminus\{i+1\}}) \oplus 2P(\Sigma_{i+1} \otimes S_{N\setminus\{i+1\}})$

e) $\Lambda_i \otimes S_N \cong P((\Delta\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\Gamma\sigma)_i \otimes S_{N\setminus\{i\}})$

f) $\Delta_i \otimes S_N \cong P((\Lambda\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\Delta\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 4P((\Gamma\sigma)_i \otimes S_{N\setminus\{i\}})$

g) $\Sigma_i \otimes S_N \cong P(\sigma_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\mu\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 6P((\Psi\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\Psi\sigma)_i(\Gamma\sigma)_{i+1} \otimes S_{N\setminus\{i,i+1\}}) \oplus 2P((\Psi\sigma)_i\Sigma_{i+1} \otimes S_{N\setminus\{i,i+1\}}) \oplus 16P(\Sigma_{i+1} \otimes S_{N\setminus\{i+1\}}) \oplus 8P((\Psi\sigma)_{i+1} \otimes S_{N\setminus\{i+1\}}) \oplus 2P((\Delta\sigma)_{i+1} \otimes S_{N\setminus\{i+1\}}) \oplus 24P((\Gamma\sigma)_{i+1} \otimes S_{N\setminus\{i+1\}}) \oplus 2P(\Gamma_{i+1} \otimes S_{N\setminus\{i+1\}}) \oplus 64S_N$

h) $\sigma_i \otimes S_N \cong P(\Sigma_i \otimes S_{N\setminus\{i\}})$

i) $(\Theta\sigma)_i \otimes S_N \cong P(\Gamma_i \otimes S_{N\setminus\{i\}})$

j) $(\Gamma\sigma)_i \otimes S_N \cong P(\Theta_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Lambda_i \otimes S_{N\setminus\{i\}}) \oplus 4P(\Delta_i \otimes S_{N\setminus\{i\}}) \oplus 4P(\Gamma_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Gamma_i\Sigma_{i+1} \otimes S_{N\setminus\{i,i+1\}})$

k) $(\mu\sigma)_i \otimes S_N \cong P(\Psi_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Sigma_i \otimes S_{N\setminus\{i\}})$

l) $(\Psi\sigma)_i \otimes S_N \cong P(\mu_i \otimes S_{N\setminus\{i\}}) \oplus 6P(\Sigma_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Sigma_i(\Gamma\sigma)_{i+1} \otimes S_{N\setminus\{i,i+1\}}) \oplus 2P(\Sigma_i\Sigma_{i+1} \otimes S_{N\setminus\{i,i+1\}})$

m) $(\Lambda\sigma)_i \otimes S_N \cong P(\Delta_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Gamma_i \otimes S_{N\setminus\{i\}})$

n) $(\Delta\sigma)_i \otimes S_N \cong P(\Lambda_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Delta_i \otimes S_{N\setminus\{i\}}) \oplus 4P(\Gamma_i \otimes S_{N\setminus\{i\}})$

PROOF

The arguments are similar to those of version A. ∎

COROLLARY 6.2.3A. *$(G = D_4.)$ Let $T \subseteq N = \{0, \dots, n-1\}$ and let $i \in T$. Let A denote any of the symbols $\Theta^{abc}, \mu, \Lambda^{abc}, \Delta^{abc}, \Psi$. Then $Head(A_i \otimes S_T)$ has no constituent of the form $\Theta^{a}_{I_1} \otimes \cdots \otimes \Gamma^{c}_{I_{14}} \otimes S_R$ with $|I_1 \cup \cdots \cup I_{14}| > 1$. If A denotes any of the symbols Γ^{abc}, then $Head(A_i \otimes S_T)$ has no constituent of the above form with $|I_1 \cup \cdots \cup I_{14}| > 2$.*

COROLLARY 6.2.3B. *$(G = A_4.)$ Let $T \subseteq N = \{0, \dots, n-1\}$ and let $i \in T$. Let A denote any of the symbols $\Theta, \Theta^*, \Lambda, \Lambda^*, \mu, \Delta, \Delta^*, \Sigma, \Sigma^*, \Psi, \Upsilon, \Upsilon^*$. Then $Head(A_i \otimes S_T)$ has no constituent of the form $\Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}} \otimes S_R$ with $|I_1 \cup \cdots \cup I_{14}| > 1$. If A denotes any of the symbols Γ, Γ^*, then $Head(A_i \otimes S_T)$ has no constituent of the above form with $|I_1 \cup \cdots \cup I_{14}| > 2$.*

COROLLARY 6.2.3C. *$(G = B_4.)$ Let $T \subseteq N = \{0, \dots, n-1\}$ and let $i \in T$. Let A denote any of the symbols $\Theta, \mu, \Lambda, \Delta, \Psi, \sigma, \Theta\sigma, \mu\sigma, \Lambda\sigma, \Delta\sigma$. Then $Head(A_i \otimes S_T)$ has no constituent of the form $\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R$ with $|I_1 \cup \cdots \cup I_{14}| > 1$. If A denotes any of the symbols $\Sigma, \Psi\sigma, \Gamma, \Gamma\sigma$, then $Head(A_i \otimes S_T)$ has no constituent of the above form with $|I_1 \cup \cdots \cup I_{14}| > 2$.*

PROOF. Suppose M is a G-module with $Head_G(M)$ having a summand of the above form with $|I_1 \cup \cdots \cup I_{14}| = m$, for some $m \in \mathbf{N}$. Let U be any subset of $N = \{0, \ldots, n-1\}$. Then by induction on $|U|$, we have that $Head_G(M \otimes S_U)$ has a summand of the above form with $|I_1 \cup \cdots \cup I_{14}| = m$. (Choose $k \in U$, and consider the separate cases $k \in R$, $k \in I_1 \cup \cdots \cup I_{14}$, and $k \notin I_1 \cup \cdots \cup I_{14} \cup R$.) Thus, $Head_G(M \otimes S_U)$ has a summand of the given form with $|I_1 \cup \cdots \cup I_{14}| = m$. Now apply Lemma 6.2.2.

∎

COROLLARY 6.2.3A. *($G = D_4$.) Assume $n > 3$. Let $T \subseteq N = \{0, \ldots, n-1\}$ and let $i \in T$. Let A and A' each denote any of the symbols $\Theta^{abc}, \mu, \Lambda^{abc}, \Delta^{abc}$. Then $Head(A_i A'_{i+1} \otimes S_T)$ has no constituent of the form $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R$ with $|I_1 \cup \cdots \cup I_{14}| > 2$.*

COROLLARY 6.2.3B. *($G = A_4$.) Assume $n > 3$. Let $T \subseteq N = \{0, \ldots, n-1\}$ and let $i \in T$. Let A and A' each denote any of the symbols $\Theta, \Theta^*, \Lambda, \Lambda^*, \mu, \Delta, \Delta^*, \Sigma, \Sigma^*, \Psi$. Then $Head(A_i A'_{i+1} \otimes S_T)$ has no constituent of the form $\Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}} \otimes S_R$ with $|I_1 \cup \cdots \cup I_{14}| > 2$.*

COROLLARY 6.2.3C. *($G = B_4$.) Assume $n > 4$. Let $T \subseteq N = \{0, \cdots, n-1\}$ and let $i \in T$. Let A and A' each denote any of the symbols $\Theta, \mu, \Lambda, \Delta, \sigma, \Theta\sigma, \mu\sigma, \Lambda\sigma, \Delta\sigma$. Then $Head(A_i A'_{i+1} \otimes S_T)$ has no constituent of the form $\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R$ with $|I_1 \cup \cdots \cup I_{14}| > 2$.*

PROOF.

Let $G = D_4$.

$$\dim_k(\mathrm{Hom}_{kG(n)}(A_i A'_{i+1} \otimes S_N, \Theta^a_{I_1} \otimes \Theta^b_{I_2} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$$

$$= \dim_k(\mathrm{Hom}_{kG(n)}(S_N, A^*_i(A')^*_{i+1} \otimes (\Theta^a_{I_1} \otimes \Theta^b_{I_2} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R))$$

$=$ multiplicity of S_N as a composition factor of $A_i^*(A')_{i+1}^* \otimes (\Theta_{I_1}^a \otimes \Theta_{I_2}^b \otimes \cdots \otimes \Gamma_{I_{14}}^c \otimes S_R)$. We consider the filtration factors of $[A_i^*(A')_{i+1}^* \otimes B_i] \otimes (\Theta_{I_1}^a \otimes \cdots \otimes B_{U\setminus\{i\}} \otimes \cdots \otimes \Gamma_{I_{14}}^c \otimes S_R)$ resulting from composition factors of $A_i^*(A')_{i+1}^* \otimes B_i$ where B is a restricted module type (possibly $B = k$). By 2^n-restricted mass considerations, the module $[A^*(A')_1^*] \otimes B$ is 2^4 restricted, i.e., the composition factors of $A_i^*(A')_{i+1}^* \otimes B_i$ are of the form $\Pi_i \Omega_{i+1} \Omega'_{i+2} \Omega''_{i+3}$. By Lemma 6.1.3, with the assumption that $n > 3$, we see that S_N cannot be a composition factor of the resulting filtration factor unless $\Pi = S$. Because of the assumptions on A, we then have that, in fact, $A_i^* \otimes B_i \cong S_i$. Thus, the same argument can then be applied to $(A')_{i+1}^* \otimes (\Theta_{I_1}^a \otimes \Theta_{I_2}^b \otimes \cdots \otimes B_{U\setminus\{i\}} \otimes \cdots \otimes \Gamma_{I_{14}}^c \otimes S_{R\cup\{i\}})$. This will show that we must have $R = N \setminus \{i, i+1\}$. The proof for the case $G = A_4$ is similar. If $G = B_4$, the 2^n-restricted mass considerations only guarantee that the module $[A^*(A')_1^*] \otimes B$ is 2^5 restricted, so that the assumption $n > 4$ is necessary in order to apply Lemma 6.1.3.

§6.3. Reduction of the Problem

We show that the 1-cohomology groups for the finite groups vanish in a large number of cases. The following lemma is a generalization of Alperin's induction step ([1]) which follows easily from the long exact sequence of cohomology for $kG(n)$.

Lemma 6.3.1 *Let D be any $kG(n)$-module, let A, B be simple $kG(n)$-modules, and let E be any simple quotient of $B \otimes D$. Let $X(A,B)$ denote the (unique up to isomorphism) $kG(n)$-module with head isomorphic to A, and radical isomorphic to a direct sum of $d = \dim_k(Ext^1_{kG(n)}(A, B))$ copies of B. Then surjectivity of the natural map*

$$Hom_{kG(n)}(A \otimes D, E) \longrightarrow Hom_{kG(n)}(X(A,B) \otimes D, E)$$

implies that $\dim_k(Ext^1_{kG(n)}(A, B)) \leq \dim_k(Ext^1_{kG(n)}(A \otimes D, E))$. ∎

In our applications, we will prove surjectivity by showing $\mathrm{Hom}_{kG(n)}(X(A,B) \otimes D, E) = 0$. In most cases we can simply check that A is not a composition factor of $D^* \otimes E$.

LEMMA 6.3.2. *Let I, J be subsets of $N = \{0, 1, \ldots, n-1\}$ with $I \neq J$. If $G = A_4$, or D_4, suppose furthermore that either*

i) $|I \Delta J| > 1$, or

ii) $|I \Delta J| = 1$, and $I \cup J = (I \cap J) \cup \{l\}$ where $l - 1 \in I \cap J$.

Then $\mathrm{Ext}^1_{kG(n)}(S_I, S_J) = 0$.

PROOF

A)(for D_4.) We may assume $J \subsetneq I$ (as S is self dual). Let $k \in N \setminus I$. We prove that $\dim_k(\mathrm{Ext}^1_{kG(n)}(S_I, S_J)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{I \cup \{k\}}, S_{J \cup \{k\}}))$ using Lemma 6.3.1, it suffices to show that

$$\mathrm{Hom}_{kG(n)}(X(S_I, S_J) \otimes S_k, S_J \otimes S_k)$$

$$(\cong \mathrm{Hom}_{kG(n)}(X(S_I, S_J), (S_k \otimes S_k) \otimes S_J)) \cong 0$$

(The result then follows by downward induction on $|I|$, as S_N is projective.)

Since $m(S) = 14$, we need to consider only those filtration factors of $(S_k \otimes S_k) \otimes S_J$ resulting from composition factors of $S_k \otimes S_k$ with mass ≥ 14 (i.e., S_k, $\Psi_k \mu_{k+1}$, $S_k \Theta^{abc}_{k+1}$, $\Psi_k \Lambda^{abc}_{k+1}$, $\mu_k \Psi_{k+1}$, $S_k \mu_{k+1}$, $\Psi_k \Delta^{abc}_{k+1}$, $S_k \Lambda^{abc}_{k+1}$, $\Psi_k \Psi_{k+1}$, $\mu_k \Gamma^{abc}_{k+1}$, and S_{k+1},) to show using Lemma 6.1.1 that S_I is not a composition factor of $(S_k \otimes S_k) \otimes S_J$. With the exception of S_{k+1}, all of the composition factors of $S_k \otimes S_k$ with mass greater than or equal to 14 are of the form $\Pi_k \Omega_{k+1}$, with $\Pi \neq k$. By Lemma 6.1.3, S_I cannot be a composition factor of $\Omega_{k+1} \otimes (\Pi_k \otimes S_J)$, otherwise, we would have

$$(2^n - 1)\theta \leq (m(\Pi) - \theta) + 2m(\Omega)$$

However, we have $m(\Pi) + 2m(\Omega) = m_{2^n}(\Pi\Omega_1) \leq m_{2^n}(S \otimes S) \leq 2m_{2^n}(S) = 28$, but we are assuming $n > 3$. This reduces us to considering the filtration factor $(S_{k+1}) \otimes S_J$. If simple, it cannot be isomorphic to S_I by the hypothesis of the lemma. $(k+1) - 1 = k \notin I \cap J$, by assumption. If it is not simple, the "only if" assertion of Lemma 6.1.1 implies that it has mass less than $m(S_{k+1}) + m(S_J) \leq m(S_I)$.

B)(for B_4.) We proceed by showing

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_I, S_J)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_I \otimes \sigma_k, S_J \otimes \sigma_k))$$

$$\leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{I\cup\{k\}}, S_{J\cup\{k\}}))$$

for arbitrary $k \in N \setminus I$, using $S_k \cong \sigma_k \otimes \Sigma_k$. The first inequality will follow from Lemma 6.3.1, if we can show that

$$\mathrm{Hom}_{kG(n)}(X(S_I, S_J) \otimes \sigma_k, S_J \otimes \sigma_k) \; (\cong \mathrm{Hom}_{kG(n)}(X(S_I, S_J), (\sigma_k \otimes \sigma_k) \otimes S_J)) \cong 0$$

This is immediate, as $m((\sigma_k \otimes \sigma_k) \otimes S_J) \leq m(\sigma_k \otimes \sigma_k) + m(S_J) = 9 + 25|J| < m(S_I)$. The second inequality will follow if we can show that $\mathrm{Hom}_{kG(n)}(X(S_I \otimes \sigma_k, S_J \otimes \sigma_k) \otimes \Sigma_k, (S_J \otimes \sigma_k) \otimes \Sigma_k) \cong \mathrm{Hom}_{kG(n)}(X(S_I \otimes \sigma_k, S_J \otimes \sigma_k), (\Sigma_k \otimes \Sigma_k) \otimes (S_J \otimes \sigma_k)) \cong 0$. Here we observe that all of the composition factors of $\Sigma_k \otimes \Sigma_k$ have mass less than 25, except for $\Sigma_k\sigma_{k+1}, \Sigma_k\mu_{k+1}, \Psi_k(\mu\sigma)_{k+1}, \Sigma_k(\Theta\sigma)_{k+1}, \Sigma_k\Lambda_{k+1}$, and $\Psi_k\Psi_{k+1}$. However, all of these composition factors are of the form $\Pi_k\Omega_{k+1}$, with $\Pi = \Sigma$ or Ψ. (In particular, $\Pi \not\cong k$.) Thus, we can apply an argument similar to the one used in (A) above. If $S_I \otimes \sigma_k$ were a composition factor of $\Omega_{k+1} \otimes ((\Pi\sigma)_k \otimes S_J)$, then by Lemma 6.1.3, we would have

$$(2^n - 1)\theta \leq (m(\Pi\sigma) - m(\sigma)) + 2m(\Omega)$$

However, we have $m(\Pi) + 2m(\Omega) = m_{2^n}(\Pi\Omega_1) \leq m_{2^n}(\Sigma \otimes \Sigma) \leq 2m_{2^n}(\Sigma) = 40$, but we are assuming $n > 3$ The result follows (as in A) by the obvious downward induction. ∎

Lemma 6 3.3 *Given disjoint subsets $I_1, \ldots, I_{14}$, and R, with $I_1 \cup \cdots \cup I_{14} \cup R \subsetneq T$ for some subset $T \subseteq N = \{0, 1, \ldots, n-1\}$, with at least one of $I_1, \ldots, I_{14}$ nonempty, then*

$$Ext^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R) = 0,$$

if $G = D_4$, and

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R) = 0,$$

if $G = B_4$

Proof A) By Lemma 6 3.1, it will suffice to show that S_T is not a composition factor of $(S_k \otimes S_k) \otimes \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R$, for $k \in N \setminus T$ We consider the filtration that results from a composition series of $S_k \otimes S_k$ Under the assumptions, $m(\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R) + 14 < m(S_T)$ Now, we observe from Table 2-1 that the composition factors of $S_k \otimes S_k$ of mass greater than 14 are all of the form $\Pi_k\Omega_{k+1}$ with $\Pi \neq k$ Thus we can apply Lemma 6.1 3 to show that S_T is not a composition factor of any of the resulting filtration factors. Otherwise, we would have

$$(2^n - 1)\theta \leq (m(\Pi) - 0) + 2m(\Omega)$$

However, we have $m(\Pi) + 2m(\Omega) = m_{2^n}(\Pi\Omega_1) \leq m_{2^n}(S \otimes S) \leq 2m_{2^n}(S) = 28$, but we are assuming $n > 3$

B) We show that

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R))$$

$$\leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_T \otimes \Sigma_k, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R \otimes \Sigma_k))$$

$$\leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{T\cup\{k\}}, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_{R\cup\{k\}}))$$

for $k \in N \setminus T$. The first inequality will follow from Lemma 6.3.1 if we can show that S_T is not a composition factor of $(\Sigma_k \otimes \Sigma_k) \otimes \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R$. The argument is similar to the one used in (A). The composition factors of $\Sigma_k \otimes \Sigma_k$ of mass greater than 25 are all of the form $\Pi_k \Omega_{k+1}$ with $\Pi \neq k$. Thus we can apply Lemma 6.1.3 to show that S_T is not a composition factor of any of the resulting filtration factors. Otherwise, we would have

$$(2^n - 1)\theta \leq (m(\Pi) - 0) + 2m(\Omega)$$

However, we have $m(\Pi) + 2m(\Omega) = m_{2^n}(\Pi\Omega_1) \leq m_{2^n}(\Sigma \otimes \Sigma) \leq 2m_{2^n}(\Sigma) = 40$, but we are assuming $n > 3$. To establish the second inequality, we show that $S_T \otimes \Sigma_k$ is not a composition factor of $(\sigma_k \otimes \sigma_k) \otimes \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R \otimes \Sigma_k$. This follows immediately from

$$m((\sigma_k \otimes \sigma_k) \otimes \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R \otimes \Sigma_k)$$

$$\leq m(\sigma_k \otimes \sigma_k) + m(\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R) + m(\Sigma_k)$$

$$\leq 9 + [25(|T| - 2) + 21] + 20 = 25|T| < m(S_T \otimes \Sigma_k)$$

LEMMA 6.3.4 *Let $I_1, \ldots, I_{14}$, and R be disjoint subsets of N, and let $T \subseteq N$. If $|T| > |R| + 2$, or if $|T \setminus (I_1 \cup \cdots \cup I_{14} \cup R)| > 1$, then*

$$Ext^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R) = 0,$$

if $G = D_4$, and

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma^*)_{I_{14}} \otimes S_R) = 0,$$

if $G = A_4$. If $G = B_4$, and if $|T \setminus (I_1 \cup \cdots \cup I_{14} \cup R)| > 1$, then

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R) = 0$$

PROOF. Let $G = D_4$. We may assume $R \subseteq T$, as S is self-dual. Suppose $|T| > |R| + 2$. Choose $k \in N \setminus T$. Suppose $k \in I_1$. Then $S_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$ has a quotient isomorphic to $\Theta^a_{I\setminus\{k\}} \otimes \cdots \otimes \Gamma^a_{I_{12}\cup\{k\}} \otimes \cdots \otimes S_R$. Now, S_T is not a composition factor of $S_k \otimes (\Theta^a_{I\setminus\{k\}} \otimes \cdots \otimes \Gamma^a_{I_{12}\cup\{k\}} \otimes \cdots \otimes S_R)$ by Lemma 6.2.1, since $|T| > |R| + 2$. Thus, Lemma 6.3.1 implies

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R))$$

$$\leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{T\cup\{k\}}, \Theta^a_{I\setminus\{k\}} \otimes \cdots \otimes \Gamma^a_{I_{12}\cup\{k\}} \otimes \cdots \otimes S_R))$$

We argue similarly if $k \in I_2 \cup \cdots \cup I_{14}$. If $k \notin I_1 \cup \cdots \cup I_{14}$, then $S_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$ is irreducible, and $(S_k \otimes S_k) \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$ has no composition factor isomorphic to S_T, by the final assertion of Lemma 6.2.1. Thus,

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R))$$

$$\leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{T\cup\{k\}}, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_{R\cup\{k\}}))$$

by Lemma 6.3.1. The result now follows by downward induction on $|T|$.

We argue somewhat differently if we are assuming only that $|T \setminus (I_1 \cup \cdots \cup I_{14} \cup R)| > 1$. Choose $k \in N \setminus T$. Then $S_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$ has a quotient isomorphic to $\Theta^a_{I_1} \otimes \cdots \otimes A_{U\setminus\{k\}} \otimes \cdots \otimes \tilde{A}_{V\cup\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R$, for some restricted module type A (possibly $A = k$ and $\tilde{A} = S$). By twisting, we may assume that $n-1 \in T \setminus (I_1 \cup \cdots \cup I_{14} \cup R)$, and that there exists $j \in T \setminus (I_1 \cup \cdots \cup I_{14} \cup R)$ with $0 \leq k < j < n-1$. Then,

$$m_{2^n}(S_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes A_{U\setminus\{k\}} \otimes \cdots \otimes \tilde{A}_{V\cup\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R))$$

$$\begin{aligned}
&\leq m_{2^n}(S_k \otimes A_k) + m_{2^n}(\Theta^a_{I_1} \otimes \cdots \otimes A_{U\setminus\{k\}} \otimes \cdots \otimes \tilde{A}_{V\cup\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R) \\
&\leq 2^k \cdot (2m(S)) + \sum_{i \in I_1 \cup \cdots \cup U\setminus\{k\} \cup \cdots \cup V \cup \{k\} \cup \cdots \cup R} 2^i \cdot m(S) \\
&\leq 2^j \cdot m(S) + \sum_{i \in I_1 \cup \cdots \cup U\setminus\{k\} \cup \cdots \cup V \cup \{k\} \cup \cdots \cup R} 2^i \cdot m(S) \\
&\leq \sum_{i=0}^{n-2} 2^i \cdot m(S) = (2^{n-1} - 1) \cdot m(S) < 2^{n-1} \cdot m(S) < m_{2^n}(S_T)
\end{aligned}$$

Therefore S_T is not a composition factor of $S_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes A_{U\setminus\{k\}} \otimes \cdots \otimes \tilde{A}_{V\cup\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$. The result then follows as in A) by Lemma 6.3.1 and downward induction on $|T|$. The arguments for the case $G = A_4$ and the proof of the claim for $G = B_4$ are similar to the above arguments. ∎

Corollary 6.3.5. *Let $I_1, \ldots, I_{14}$, and T be disjoint subsets of N with $|T| > 1$, then*

$$Ext^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}) = 0,$$

if $G = D_4$,

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma^*)_{I_{14}}) = 0,$$

if $G = A_4$, and

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}) = 0,$$

if $G = B_4$.

For later use, we still need to the strengthen the result of Lemma 6.3.4 if $G = B_4$.

LEMMA 6.3.6. *Let $G = B_4$. Let $I_1, \ldots, I_{14}$, and R be disjoint subsets of N, and let $T \subseteq N$, with $I_1 \cup \cdots \cup I_{14} \cup R \subseteq T$. Suppose that $|T| > |R| + 3$, or that $|T| = |R| + 3$ and the 3 elements of $T \setminus R$ are not consecutive (modulo n). Then*

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R) = 0$$

PROOF. Let $k \in N \setminus T$. By the hypothesis of the lemma, there exists $i \in T \setminus R$ such that $[i] \notin \{[k], [k+1], [k+2], [k+3]\}$, where $[\ ]$ indicates congruence class modulo n. Therefore, by twisting, if necessary, and applying Lemma 6.1.3, S_T cannot be a composition factor of $(S_k \otimes S_k) \otimes (\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R)$, otherwise,

$$(2^n - 1)\theta \leq (m(B_i) - m(S_i)) + 2^{n-4} m_{2^n}(S \otimes S)$$

$$\leq -\theta + 2^{n-4} m_{2^n}(S \otimes S),$$

for some $B \in \{k, \Theta, \mu, \Lambda, \ldots, \Gamma\sigma\}$ since $n - 4 \geq$ the least nonnegative residue of $k - i$ mod (n). Thus, we would have

$$2^4\theta \leq m_{2^n}(S \otimes S) = 2 \cdot 25 = 50,$$

but $\theta = 4$.

Lemma 6.3.7A *Let $G=D_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of $N = \{0, 1, \ldots, n-1\}$ with $|I_1 \cup \cdots \cup I_{14}| > 2$. Let T be an arbitrary subset of N, and let*

$$X = X(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}})$$

be as in the notation of Lemma 6.3.1. Let B denote one of $\Theta^a, \ldots, \Gamma^c$, and let U denote the corresponding subscript (i.e., U is one of $I_1, \ldots, I_{14}$.) Suppose that $k \in U$ and that $\Pi_k \otimes \Omega_j$ is a composition factor of a module of the form $A_k \otimes \tilde{B}_k$, where A is one of $\Theta^a, \ldots, \Gamma^c, S$, where $\tilde{B}$ is one of $\Theta^a, \ldots, \Gamma^c$, where Ω is one of $k, \Theta^{abc}, \mu, \Lambda^{abc}, \Delta^{abc}, \Psi$, where $j \in \{k+1, k+2\}$, and where Π is any of $k, \Theta^{abc}, \ldots, \Gamma^{abc}, S$, with $\Pi \neq B$. Then

$$Hom_{kG(n)}(X, \Pi_k \otimes \Omega_j \otimes (\Theta^a_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})) = 0$$

Proof We have

$$\dim_k(\mathrm{Hom}_{kG(n)}(X, \Pi_k \otimes \Omega_j \otimes (\Theta^a_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})))$$

$$\leq \dim_k(\mathrm{Hom}_{kG(n)}(S_T \otimes \Omega^*_j, \Pi_k \otimes \Theta^a_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}}))$$

$$+d \cdot \dim_k(\mathrm{Hom}_{kG(n)}(\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}, \Omega_j \otimes (\Pi_k \otimes \Theta^a_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})))$$

The first term is zero by Corollary 6.2.3, by the assumption on Ω, since $|I_1 \cup \cdots \cup I_{14}| > 2$. The second term is handled by Lemma 6.1.3, because of the assumption $\Pi \neq B$. If $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}$ were a composition factor of $\Omega_j \otimes (\Pi_k \otimes \Theta^a_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})$, we would have

$$(2^n - 1)\theta \leq (m(\Pi) - m(B)) + 2^{j-k} m(\Omega),$$

$$\leq m_{2^n}(\Pi\Omega_{j-k}) - m(B) \leq m_{2^n}(A \otimes \tilde{B}) - m(B)$$

$$\leq m(A) + m(\tilde{B}) - m(B) \leq 14 + 8,$$

but we are assuming that $n > 3$.

Lemma 6.3.7B. *Let $G = A_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of $N = \{0, 1, \ldots, n-1\}$ with $|I_1 \cup \cdots \cup I_{14}| > 2$. Let T be an arbitrary subset of N, and let*

$$X = X(S_T, \Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}})$$

be as in the notation of Lemma 6.3.1. Let B denote one of $\Theta, \ldots, \Gamma^*$, and let U denote the corresponding subscript (i.e., U is one of $I_1, \ldots, I_{14}$.) Suppose that $k \in U$ and that $\Pi_k \otimes \Omega_j$ is a composition factor of a module of the form $A_k \otimes \hat{B}_k$, where A is one of $\Theta, \ldots, \Gamma^*, S$, where $\hat{B}$ is one of $\Theta, \ldots, \Gamma^*$, where Ω is one of $k, \Theta, \Theta^*, \mu, \Lambda, \Lambda^*, \Delta, \Delta^*, \Sigma, \Sigma^*, \Psi, \Upsilon, \Upsilon^*$, where $j \in \{k+1, k+2\}$, and where Π is any of the restricted module types $k, \Theta, \ldots, \Gamma^*, S$, with $\Pi \neq B$. Then

$$Hom_{kG(n)}(X, \Pi_k \otimes \Omega_j \otimes (B_{I_1} \otimes \cdots \otimes B_{U \setminus \{k\}} \otimes \cdots \otimes \Gamma^*_{I_{14}})) = 0$$

Proof. The proof is virtually identical to that of Lemma 6.3.7A.

Lemma 6.3.7C. *Let $G = B_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of $N = \{0, 1, \ldots, n-1\}$ with either $|I_1 \cup \cdots \cup I_{14}| > 3$, or $|I_1 \cup \cdots \cup I_{14}| = 3$ such that the elements of $I_1 \cup \cdots \cup I_{14}$ are not all consecutive. Let T be an arbitrary subset of N, and let*

$$X = X(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$$

be as in the notation of Lemma 6.3.1. Let B denote one of $\Theta, \ldots, \Gamma\sigma$, and let U denote the corresponding subscript (i.e., U is one of $I_1, \ldots, I_{14}$.) Suppose that $k \in U$ and that $\Pi_k \Omega_{k+1} \Omega'_{k+2}$ is a composition factor of a module of the form $A_k \otimes \hat{B}_k$, where A is one of $\Theta, \ldots, \Gamma\sigma, S$, where $\hat{B}$ is one of $\Theta, \ldots, \Gamma\sigma$, and where Π is any of the restricted module types $k, \Theta, \ldots, \Gamma\sigma, S$, with $\Pi \neq B$. Then

$$Hom_{kG(n)}(X, \Pi_k \Omega_{k+1} \Omega'_{k+2} \otimes (\Theta_{I_1} \otimes \cdots \otimes B_{U \setminus \{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})) = 0$$

Proof We have

$$\dim_k(\mathrm{Hom}_{kG(n)}(X, \Pi_k\Omega_{k+1}\Omega'_{k+2} \otimes (\Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})))$$

$$\leq \dim_k(\mathrm{Hom}_{kG(n)}(S_T \otimes \Omega^*_{k+1}(\Omega')^*_{k+2}, \Pi_k \otimes \Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}))$$

$$+d \cdot \dim_k(\mathrm{Hom}_{kG(n)}(\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}},$$

$$\Omega_{k+1}\Omega'_{k+2} \otimes (\Pi_k \otimes \Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})))$$

New, it is easily checked using 2^n restricted mass and the assumption on $\bar{B}$, (or by direct inspection of Table 2-3) that Ω' is never equal to $\Sigma, \Psi, \Psi\sigma, \Gamma$ or $\Gamma\sigma$, and that if $\Omega = \Sigma, \Psi\sigma, \Gamma$ or $\Gamma\sigma$, then $\Omega' = k$. By inspection of Table 2-3, it is checked that if $\Omega = \Psi$, then $\Omega' = k$. (Also, it is obvious that neither Ω nor Ω' are ever of type S.) Therefore, the first term is zero by Corollary 6.2.3, because of the assumptions on $I_1 \cup \cdots \cup I_{14}$. The second term is handled by Lemma 6.1.3, because of the assumption $\Pi \neq B$. If $\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}$ were a composition factor of $\Omega_{k+1}\Omega'_{k+2} \otimes (\Pi_k \otimes \Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$, we would have

$$(2^n - 1)6 \leq (m(\Pi) - m(B)) + 2m(\Omega) + 4m(\Omega'),$$

$$\leq m_{2^n}(\Pi\Omega_1\Omega'_2) - m(B) \leq m_{2^n}(A \otimes \bar{B}) - m(B)$$

$$\leq m(A) + m(\bar{B}) - m(B) \leq 25 + 21 - 4,$$

but we are assuming that $n > 3$.

Lemma 6.3.8A *Let $G = D_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of N with $|I_1 \cup \cdots \cup I_{14}| > 2$. Let T be an arbitrary subset of N, and let*

$$X = X(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}})$$

be as in the notation of Lemma 6.3.1. Let A denote one of $k, \Theta^{abc}, \mu, \Lambda^{abc}, \Delta^{abc}, \Psi$, and let $k \in N$. Then

$$Hom_{kG(n)}(X, A_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}})) = 0$$

Proof. We induct on the quantity $m(A)$, the result being obvious for $m(A) = 0$. If $A_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}})$ is simple, then it is not isomorphic to S_T, because of the assumption $|I_1 \cup \cdots \cup I_{14}| > 2$. Otherwise, it can be written in the form $(A_k \otimes B_k) \otimes (\Theta^a_{I_1} \otimes \ldots \otimes B_{U \setminus \{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})$. Because of the assumption on A, the composition factors of $A_k \otimes B_k$ are all of the form $\Pi_k \otimes \Omega_j$, with $j \in \{k+1, k+2\}$ and $\Omega \in \{k, \Theta^{abc}, \mu, \Lambda^{abc}, \Delta^{abc}, \Psi\}$ (cf. Table 2-1.) If $\Pi \neq B$, then we may apply Lemma 6.3.7A with $\tilde{B} = B$. If $\Pi = B$, then we must have $m(\Omega) < m(A)$ by the "only if" assertion of Lemma 6.1.1, so that induction may be applied.

Lemma 6.3.8B. *Let $G = A_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of N with $|I_1 \cup \cdots \cup I_{14}| > 2$. Let T be an arbitrary subset of N, and let*

$$X = X(S_T, \Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}})$$

be as in the notation of Lemma 6.3.1. Let A denote one of $k, \Theta, \Theta^*, \mu, \Lambda, \Lambda^*, \Delta, \Delta^*, \Sigma, \Sigma^*, \Psi, \Upsilon, \Upsilon^*$, and let $k \in N$. Then

$$Hom_{kG(n)}(X, A_k \otimes (\Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}})) = 0.$$

Proof. The proof is almost identical to that of Lemma 6.3.8A.

Lemma 6.3.8C. *Let $G = B_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of N with either $|I_1 \cup \cdots \cup I_{14}| > 3$, or $|I_1 \cup \cdots \cup I_{14}| = 3$ such that the elements of $I_1 \cup \cdots \cup I_{14}$ are not all consecutive. Let T be an arbitrary subset of N, and let*

$$X = X(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$$

be as in the notation of Lemma 6.3.1. Let A denote one of $k, \Theta, \ldots, \Gamma\sigma$, and let $k \in N$. Then

$$Hom_{kG(n)}(X, A_k \otimes (\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})) = 0.$$

Proof. We induct on the quantity $m(A)$, the result being obvious for $m(A) = 0$. If $A_k \otimes (\Theta^{\epsilon}_{I_1} \otimes \cdots \otimes \Gamma^{\epsilon}_{I_{14}})$ is simple, then it is not isomorphic to S_T, because $|I_1 \cup \cdots \cup I_{14}| > 2$. Otherwise, it can be written in the form $(A_k \otimes B_k) \otimes (\Theta_{I_1} \otimes \cdots \otimes B_{U \setminus \{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$. The composition factors of $A_k \otimes B_k$ are all of the form $\Pi_k \Omega_{k+1} \Omega'_{k+2}$. If $\Pi \neq B$, then we may apply Lemma 6.3.7C with $\bar{B} = B$. If $\Pi = B$, then we must have either $\Omega' = k$ and $m(\Omega) < m(A)$, or $\Omega = k$ and $m(\Omega') < m(A)$, by 2^n restricted mass considerations (with the assumption $A \neq S$), and by the "only if" assertion of Lemma 6.1.1. Thus, induction may be applied.

Lemma 6.3.9A. *Let $G = D_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of N with $|I_1 \cup \cdots \cup I_{14}| > 2$. Let T be an arbitrary subset of N. Suppose furthermore that either $T \not\subseteq I_1 \cup \cdots \cup I_{14}$, or that $\Theta^{\epsilon}_{I_1} \otimes \cdots \otimes \Gamma^{\epsilon}_{I_{14}}$ is not Galois conjugate to a module of the form $\mu\Gamma_1^{abc}\Gamma_2^{abc}, \mu\Gamma_1^{abc}\Theta_2^{abc}, \mu\Theta_1^{abc}\Gamma_2^{abc}$, or $\mu\Theta_1^{abc}\Theta_2^{abc}$. Then*

$$Ext^1_{kG(n)}(S_T, \Theta^{\epsilon}_{I_1} \otimes \cdots \otimes \Gamma^{\epsilon}_{I_{14}}) = 0.$$

Proof. If $|T| > 2$, we apply Lemma 6.3.4. Thus, we may assume $I_1 \cup \cdots \cup I_{14} \not\subseteq T$, let $k \in (I_1 \cup \cdots \cup I_{14}) \setminus T$ and, as in [17], proceed to find disjoint subsets $I_1, \ldots, I_{14}$ with $I'_1 \cup \cdots \cup I'_{14} = I_1 \cup \cdots \cup I_{14}$, and such that $\dim_k(\mathrm{Ext}^1_{kG(n)}(S_T, \Theta^{\epsilon}_{I_1} \otimes \cdots \otimes \Gamma^{\epsilon}_{I_{14}})) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{T \cup \{k\}}, \Theta^{\epsilon}_{I'_1} \otimes \cdots \otimes \Gamma^{\epsilon}_{I'_{14}}))$, iterating, if necessary, to reduce to the case $|T| > 2$. To accomplish this, we employ Lemma 6.3.1, as usual, and show that

$$\mathrm{Hom}_{kG(n)}(X(S_T, \Theta^{\epsilon}_{I_1} \otimes \cdots \otimes \Gamma^{\epsilon}_{I_{14}}) \otimes S_k, E) = 0,$$

where E is a simple quotient of $S_k \otimes (\Theta^a_{I_1} \otimes \ldots \otimes \Gamma^c_{I_{14}})$. We assume from the outset that $n > 3$, and use Lemma 6.1.3. Suppose $k \in U$ (where U is one of $I_1, \ldots, I_{14}$) and that B is the module type with subscript U in the above notation. Then a simple quotient of $S_k \otimes (\Theta^a_{I_1} \otimes \ldots \otimes \Gamma^c_{I_{14}})$ is given by $E \cong \Theta^a_{I_1} \otimes \ldots \otimes B_{U \setminus \{k\}} \otimes \ldots \otimes \tilde{B}_{\tilde{U} \cup \{k\}} \otimes \ldots \otimes \Gamma^c_{I_{14}}$ (for some $\tilde{B}$), with $m(\tilde{B}) - m(B) \leq 11 - 3 = 8$. To obtain the desired result, we analyze the filtration factors of $S_k \otimes E \cong (S_k \otimes \tilde{B}_k) \otimes (\Theta^a_{I_1} \otimes \ldots \otimes B_{U \setminus \{k\}} \otimes \ldots \otimes \ldots \otimes \Gamma^c_{I_{14}})$ resulting from the composition factors of $S_k \otimes \tilde{B}_k$, and show that there are no nonzero homomorphisms from $X(S_T, \Theta^a_{I_1} \otimes \ldots \otimes \Gamma^c_{I_{14}})$ into each filtration factor.

The composition factors of $S_k \otimes \tilde{B}_k$ are of the following three types:

i) $\Pi_k \Omega_j$, with $j \in \{k+1, k+2\}$ and $\Omega \neq \Gamma^{abc}, S$. (Only those with $\Omega \neq k$ are of concern because of the assumption $|I_1 \cup \ldots \cup I_{14}| > 2$.) If $\Pi = B$, we may apply Lemma 6.3.8A. Otherwise we apply Lemma 6.3.7A.

ii) $\Pi_k \Omega_{k+1}$, with $\Omega = \Gamma^{abc}$. This type of composition factor occurs only in the following two situations:

a) $\tilde{B} = \Psi$ (i.e., $B = \mu$). Then we have $\Pi = k \neq B$, so we may apply Lemma 6.1.3 to show that $\Theta^a_{I_1} \otimes \ldots \otimes \Gamma^c_{I_{14}}$ is not a composition factor of

$$\Gamma^{abc}_{k+1} \otimes (\Theta^a_{I_1} \otimes \ldots \otimes \mu_{I_4 \setminus \{k\}} \otimes \ldots \otimes \Gamma^c_{I_{14}})$$

This is because

$$(2^n - 1)6 > (m(k) - m(\mu)) + 2m(\Gamma^{abc})$$

as we assume $n > 3$. Thus,

$$\mathrm{Hom}_{kG(n)}(X, \Gamma^{abc}_{k+1} \otimes (\Theta^a_{I_1} \otimes \ldots \otimes \mu_{I_4 \setminus \{k\}} \otimes \ldots \otimes \Gamma^c_{I_{14}}))$$

$$\subseteq \mathrm{Hom}_{kG(n)}(S_T \otimes \Gamma^{abc}_{k+1}, \Theta^a_{I_1} \otimes \ldots \otimes \mu_{I_4 \setminus \{k\}} \otimes \ldots \otimes \Gamma^c_{I_{14}}),$$

which can be nonzero only if

$$\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4 \setminus \{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \Gamma^{abc}_{k+1}\Gamma^{abc}_{k+2},$$

(i.e., $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \mu_k \Gamma^{abc}_{k+1}\Gamma^{abc}_{k+2}$) and $T = \{k+1, k+2\}$, or if

$$\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4 \setminus \{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \Gamma^{abc}_{k+1}\Theta^{abc}_{k+2},$$

(i.e., $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \mu_k \Gamma^{abc}_{k+1}\Theta^{abc}_{k+2}$) and $T = \{k+1\}$, by Corollary 6.2.3, together with the assumption $|I_1 \cup \cdots \cup I_{14}| > 2$.

h) $\tilde{B} = \Gamma^{abc}$ (i.e., $B = \Theta^{abc}$). In this situation, we have $\Pi = B \neq k$, and thus

$$\mathrm{Hom}_{kG(n)}(S_T \otimes \Gamma^{abc}_{k+1}, \Pi_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes B_{U \setminus \{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}}))$$

$$= \mathrm{Hom}_{kG(n)}(S_T \otimes \Gamma^{abc}_{k+1}, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}) = 0$$

by Corollary 6.2.3, because $|I_1 \cup \cdots \cup I_{14}| > 2$. Thus,

$$\mathrm{Hom}_{kG(n)}(X, \Gamma^{abc}_{k+1} \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}))$$

$$\subseteq \mathrm{Hom}_{kG(n)}(\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}, \Gamma^{abc}_{k+1} \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}})).$$

Now, if $k+1 \notin I_1 \cup \cdots \cup I_{14}$, then the above hom group is obviously zero. Therefore we consider the filtration factors of $(\Gamma^{abc}_{k+1} \otimes B'_{k+1}) \otimes (\Theta^a_{I_1} \otimes \cdots \otimes B'_{V \setminus \{k+1\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})$ that result from the composition factors of $\Gamma^{abc}_{k+1} \otimes B'_{k+1}$, which are all of the form $\Pi'_{k+1}\Omega'_{k+2}\Omega''_{k+3}$ with $\Pi' \neq B'$. So, by Lemma 6.1.3, if $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}$ were a

composition factor of $(\Omega'_{k+2}\Omega''_{k+3}) \otimes (\Pi'_{k+1} \otimes \Theta^a_{I_1} \otimes \cdots \otimes B'_{V\setminus\{k+1\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})$, we would have

$$(2^n - 1)\theta \leq (m(\Pi') - m(B')) + 2m(\Omega') + 4m(\Omega'')$$

$$\leq m_{2^n}(\Gamma^{abc} \otimes B') - m(B') \leq m(\Gamma^{abc}) + m(B') - m(B') = 11,$$

which is impossible under the assumption $n > 3$.

iii) $\Pi_k\Omega_{k+1}\Omega'_{k+2}$ with $\Omega, \Omega' = \Theta^{abc}$. These composition factors occur only in the following two situations:

a) $B = \Psi$ (i.e., $B = \mu$). Here we have $\Pi = k \neq B$, so the usual argument involving Lemma 6.1.3 shows that $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}$ is not a composition factor of $(\Omega_{k+1}\Omega'_{k+2}) \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})$ if $n > 3$; otherwise we would have

$$(2^n - 1)\theta \leq (m(k) - m(\mu)) + 2m(\Omega) + 4m(\Omega')$$

$$\leq m(S) + m(\Psi) - m(\mu) = 18.$$

Thus,

$$\mathrm{Hom}_{kG(n)}(X, \Omega_{k+1}\Omega'_{k+2} \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}}))$$

$$\subseteq \mathrm{Hom}_{kG(n)}(S_T \otimes (\Omega_{k+1}\Omega'_{k+2}), \Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}}),$$

which by Lemma 6.2.2, can be nonzero only if $\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \Gamma^{abc}_{k+1}\Gamma^{abc}_{k+2}$ (i.e., only if $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \mu_k\Gamma^{abc}_{k+1}\Gamma^{abc}_{k+2}$), and if $T = \{k+1, k+2\}$, if $\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \Gamma^{abc}_{k+1}\Theta^{abc}_{k+2}$, (i.e., $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \mu_k\Gamma^{abc}_{k+1}\Theta^{abc}_{k+2}$) and $T = \{k+1\}$, if $\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \Theta^{abc}_{k+1}\Gamma^{abc}_{k+2}$, (i.e., $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \mu_k\Theta^{abc}_{k+1}\Gamma^{abc}_{k+2}$) and $T = \{k+2\}$, or if $\Theta^a_{I_1} \otimes \cdots \otimes \mu_{I_4\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \Theta^{abc}_{k+1}\Theta^{abc}_{k+2}$, (i.e., $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \mu_k\Theta^{abc}_{k+1}\Theta^{abc}_{k+2}$) and $T = \phi$

b) $\tilde{B} = \Gamma^{abc}$ (i.e., $B = \Theta^{abc}$.) Then $\mathbf{II} = B \neq k$, so in this situation, we have

$$\mathrm{Hom}_{kG(n)}(S_T \otimes (\Omega_{k+1}\Omega'_{k+2}), \Pi_k \otimes (\Theta^a_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \Gamma^c_{I_{14}}))$$

$$= \mathrm{Hom}_{kG(n)}(S_T \otimes (\Omega_{k+1}\Omega'_{k+2}), \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}})) = 0$$

by Corollary 6.2.3, so that

$$\mathrm{Hom}_{kG(n)}(X, \Omega_{k+1}\Omega'_{k+2} \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}))$$

$$\subseteq \mathrm{Hom}_{kG(n)}(\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}, \Omega_{k+1}\Omega'_{k+2} \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}))$$

We consider the filtration factors of $\Omega_{k+1}\Omega'_{k+2} \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}})$ that result from the composition factors of $\Omega_{k+1} \otimes B'_{k+1}$ for some B' (possibly $B' = k$.) Because $\Omega = \Theta^{abc}$, we have that the composition factors of $\Omega_{k+1} \otimes B'_{k+1}$ are of the form $\Pi'_{k+1}\Omega''_{k+2}$ with $\Pi' \neq B'$. So, by Lemma 6.1.3, if $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}$ were a composition factor of $(\Omega''_{k+2} \otimes \Omega'_{k+2}) \otimes (\Pi'_{k+1} \otimes \Theta^a_{I_1} \otimes \cdots \otimes B'_{V\setminus\{k+1\}} \otimes \cdots \otimes \Gamma^c_{I_{14}})$, we would have

$$(2^n - 1)\theta \leq (m(\Pi') - m(B')) + m_{2^n}(\Omega''_1 \otimes \Omega'_1)$$

$$\leq (m(\Pi') - m(B')) + m_{2^n}(\Omega''_1) + m_{2^n}(\Omega'_1)$$

$$= m_{2^n}(\Pi'\Omega''_1) + 2m(\Omega') - m(B')$$

$$\leq m_{2^n}(\Omega \otimes B') + 2m(\Omega') - m(B') \leq m(\Omega) + m(B') + 2m(\Omega') - m(B') = 9.$$

Lemma 6.3.9B. *Let $G = A_4$. Let $I_1, \ldots, I_{14}$ be disjoint subsets of N with $|I_1 \cup \cdots \cup I_{14}| > 2$. Let T be an arbitrary subset of N. Suppose furthermore that either*

$T \nsubseteq I_1 \cup \cdots \cup I_{14}$, *or that neither* $\Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}}$ *nor its dual is Galois conjugate to a module of the form* $\Lambda\Gamma_1\Gamma_2^*, \Lambda\Gamma_1\Theta_2^*, \Lambda\Theta_1\Gamma_2^*$, *or* $\Lambda\Theta_1\Theta_2^*$. *Then*

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}}) = 0.$$

PROOF. The argument is very similar to that of Lemma 6.3.8A, and even simpler in some parts. The restricted modules $\Lambda^{(*)}, \Upsilon^{(*)}$, and $\Gamma^{(*)}$ for A_4 play the roles of μ, Ψ, and Γ^{abc} of D_4, respectively. The only differences in the proof are that in parts (ii) b, and (iii) b, we have $\Pi = B^* \neq k$ (instead of $\Pi = B$). Thus, the arguments involving Corollary 6.2.3 still goes through. Also, because $\Pi \neq B$, the mass argument involving Lemma 6.1.3 goes through immediately, without further expansion of tensor products.

LEMMA 6.3.9C. *Let* $G = B_4$. *Let* $I_1, \ldots, I_{14}$ *be disjoint subsets of* N *with* $|I_1 \cup \cdots \cup I_{14}| > 3$, *or with* $|I_1 \cup \cdots \cup I_{14}| = 3$ *such that the 3 elements of* $I_1 \cup \cdots \cup I_{14}$ *are not all consecutive. Let* T *be an arbitrary subset of* N. *Then*

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}) = 0.$$

PROOF. If $I_1 \cup \cdots \cup I_{14} \subseteq T$, we apply Lemma 6.3.6. Thus, we may assume otherwise, let $k \in (I_1 \cup \cdots \cup I_{14}) \setminus T$ and find disjoint subsets $I_1', \ldots, I_{14}'$ with $I_1' \cup \cdots \cup I_{14}' = I_1 \cup \cdots \cup I_{14}$, and such that $\dim_k(\mathrm{Ext}^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{T\cup\{k\}}, \Theta_{I_1'} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}'}))$, iterating, if necessary, to reduce to the case $I_1 \cup \cdots \cup I_{14} \subseteq T$. To accomplish this, we employ Lemma 6.3.1, as usual, and show that

$$\mathrm{Hom}_{kG(n)}(X(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}) \otimes S_k, E) = 0,$$

where E is a simple quotient of $S_k \otimes (\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$. We assume from the outset that $n > 3$, and use Lemma 6.1.3. Suppose $k \in U$ (where U is one of $I_1, \ldots, I_{14}$) and that B is the module type with subscript U in the above notation. Then a simple quotient of $S_k \otimes (\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$ is given by $E \cong \Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \hat{B}_{\hat{U}\cup\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}$ (for some $\hat{B}$), with $m(\hat{B}) - m(B) \leq 21 - 4 = 17$. To obtain the desired result, we analyze the filtration factors of $S_k \otimes E \cong (S_k \otimes \hat{B}_k) \otimes (\Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$ resulting from the composition factors of $S_k \otimes \hat{B}_k$, and show that there are no nonzero homomorphisms from $X(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$ into each filtration factor.

The composition factors of $S_k \otimes B_k$ are of the following three types:

i) $\Pi_k\Omega_j$, with $j \in \{k+1, k+2\}$ and $\Omega \neq S$. (Only those with $\Omega \neq k$ are of concern because of the assumptions on $I_1 \cup \cdots \cup I_{14}$.) If $\Pi = B$, we may apply Lemma 6.3.8C. Otherwise we apply Lemma 6.3.7C.

ii) $\Pi_k\Omega_{k+1}\Omega'_{k+2}$ with $\Omega, \Omega' \notin \{k, \Psi, \Psi\sigma, \Gamma, \Gamma\sigma, \Sigma, S\}$. (This follows by 2^n-restricted mass considerations except the claim $\Omega \neq \Psi$, which is verified by inspection of Table 2-3.) Because of the assumptions on $I_1 \cup \cdots \cup I_{14}$,

$$\mathrm{Hom}_{kG(n)}(S_T \otimes (\Omega_{k+1}\Omega'_{k+2}), \Pi_k \otimes \Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}) = 0$$

by Lemma 6.2.2. Thus

$$\mathrm{Hom}_{kG(n)}(X, \Omega_{k+1}\Omega'_{k+2} \otimes (\Pi_k \otimes \Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}))$$

$$\subseteq \mathrm{Hom}_{kG(n)}(\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}, \Omega_{k+1}\Omega'_{k+2} \otimes (\Pi_k \otimes \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}))$$

We now consider the following two situations:

a) $\Pi \neq B$. The usual argument involving Lemma 6.1.3 shows that $\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}$ is not a composition factor of $(\Omega_{k+1}\Omega'_{k+2}) \otimes (\Pi_k \otimes \Theta_{I_1} \otimes \cdots \otimes B_{U\setminus\{k\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$ if $n > 3$, otherwise we would have

$$(2^n - 1)\theta \leq (m(\Pi) - m(B)) + 2m(\Omega) + 4m(\Omega')$$

$$\leq m(S) + m(\bar{B}) - m(B) \leq 25 + 17$$

b) $\Pi = B$. We consider the filtration factors of $\Omega_{k+1}\Omega'_{k+2} \otimes (\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$ that result from the composition factors of $\Omega_{k+1} \otimes B'_{k+1}$ for some B' (possibly $B' = k$.) If $\Omega \neq \mu$, we have that the composition factors of $\Omega_{k+1} \otimes B'_{k+1}$ are of the form $\Pi'_{k+1}\Omega''_{k+2}\Omega'''_{k+3}$ with $\Pi' \neq B'$. (cf. Table 2-3.) So, by Lemma 6.1.3, if $\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}}$ were a composition factor of $(\Omega''_{k+2}\Omega'''_{k+3} \otimes \Omega'_{k+2}) \otimes (\Pi'_{k+1} \otimes \Theta_{I_1} \otimes \cdots \otimes B'_{V\setminus\{k+1\}} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$, we would have

$$(2^n - 1)\theta \leq (m(\Pi') - m(B')) + m_{2^n}(\Omega''_1\Omega'''_2 \otimes \Omega'_1)$$

$$\leq (m(\Pi') - m(B')) + m_{2^n}(\Omega''_1\Omega'''_2) + m_{2^n}(\Omega'_1)$$

$$= m_{2^n}(\Pi'\Omega''_1\Omega'''_2) + 2m(\Omega') - m(B')$$

$$\leq m_{2^n}(\Omega \otimes B') + 2m(\Omega') - m(B') \leq m(\Omega) + m(B') + 2m(\Omega') - m(B')$$

$$< (1/2)m_{2^n}(\Pi\Omega_1\Omega_2) \leq (1/2)(m(S) + m(B)) \leq 23$$

If $\Omega = \mu$, we have that the composition factors of $\Omega_{k+1} \otimes B'_{k+1}$ of the form $\Pi'_{k+1}\Omega''_{k+2}\Omega'''_{k+3}$ with $\Pi' = B'$ all have $\Omega'' = \Omega''' = k$. This follows easily from 2^n restricted mass considerations. We then consider the filtration factors of $\Omega'_{k+2} \otimes (\Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}})$ that result from the composition factors of $\Omega'_{k+2} \otimes B''_{k+2}$ for some B'' (possibly $B'' = k$.) By 2^n restricted mass considerations, we observe that we never have $\Pi = B$, $\Omega = \mu$, and $\Omega' = \mu$, so that the preceding argument involving Lemma 6.1.3 may now be applied. ∎

Corollary 6.3.10A *Let $G = D_4$. Let $I_1, \ldots, I_{14}$, and T be disjoint subsets of N with $|I_1 \cup \cdots \cup I_{14}| > 2$. Suppose that either $T \neq \phi$, or that $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}$ is not Galois conjugate to a module of the form $\mu\Theta_1^{abc}\Theta_2^{abc}$. Then*

$$Ext^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}) = 0$$

Proof. We apply the result of Lemma 6.3.9A in all cases except $T = \phi$ and $I_1 \cup \cdots \cup I_{14}$ of the form $\{k, k+1, k+2\}$. In all of the exceptional cases (of Lemma 6.3.9A) except $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \cong \mu_k\Theta^{abc}_{k+1}\Theta^{abc}_{k+2}$, we can apply the iteration argument used in the proof of Lemma 6.3.7 to show

$$\mathrm{Ext}^1_{kG(n)}(k, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}) \leq \mathrm{Ext}^1_{kG(n)}(S_k, \Theta^a_{I'_1} \otimes \cdots \otimes \Gamma^c_{I'_{14}}) \leq$$

$$\cdots \leq \mathrm{Ext}^1_{kG(n)}(S_{\{k,k+1,k+2\}}, \Theta^a_{I''_1} \otimes \cdots \otimes \Gamma^c_{I''_{14}}),$$

the arguments go through because the sets $\phi, \{k\}, \ldots, \{k, k+1, k+2\}$ are never of the form $\{k+1\}$, $\{k+2\}$ or $\{k+1, k+2\}$ (and $n > 3$ so that $k+3 \not\equiv k \bmod(n)$.)

Corollary 6.3.10B *Let $G = A_4$. Let $I_1, \ldots, I_{14}$, and T be disjoint subsets of N with $|I_1 \cup \cdots \cup I_{14}| > 2$. Suppose that either $T \neq \phi$, or that neither $\Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}}$ nor its dual is Galois conjugate to a module of the form $\Lambda\Theta_1\Theta_2^*$. Then*

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes \Gamma^*_{I_{14}}) = 0$$

Proof. The proof is almost identical to that of Lemma 6.3.10A.

Corollary 6.3.10C *Let $G = B_4$. Let $I_1, \ldots, I_{14}$, and T be disjoint subsets of N with $|I_1 \cup \cdots \cup I_{14}| > 3$, or with $|I_1 \cup \cdots \cup I_{14}| = 3$ such that the elements of $I_1 \cup \cdots \cup I_{14}$ are not all consecutive. Then*

$$Ext^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}) = 0$$

PROOF. This is just a weakened version of Lemma 6.3.9C.

LEMMA 6.3.11. *Let $I_1, \ldots, I_{14}$, and R be disjoint sets with $I_1 \cup \cdots \cup I_{14} \cup R \subseteq T$ for some set $T \subseteq N = \{0, 1, \ldots, n-1\}$, with $|I_1 \cup \cdots \cup I_{14}| \geq 2$. Suppose furthermore that there exist $i, j \in I_1 \cup \cdots \cup I_{14}$ with $|i - j| > 2$. Then*

$$Ext^1_{kG(n)}(S_T, \Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R) = 0,$$

if $G = D_4$, and

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma^*)_{I_{14}} \otimes S_R) = 0,$$

if $G = A_4$, and

$$Ext^1_{kG(n)}(S_T, \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R) = 0$$

if $G = B_4$.

PROOF.

Let $G = D_4$. Let $k \in N \setminus T$. By the hypothesis of the lemma, there exists $i \in I_1 \cup \cdots \cup I_{14}$ such that $[i] \notin \{[k], [k+1], [k+2], [k+3]\}$, where $[\cdot]$ indicates congruence class modulo n. Therefore, by Lemma 6.1.3, S_T cannot be a composition factor of $(S_k \otimes S_k) \otimes (\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}} \otimes S_R)$, otherwise,

$$(2^n - 1)\theta \leq (m(B_i) - m(S_i)) + 2^{n-4} m_{2^n}(S \otimes S)$$

$$\leq -\theta + 2^{n-4} m_{2^n}(S \otimes S),$$

since $n - 4 \geq$ the least nonnegative residue of $k - i \bmod (n)$. Thus, we would have

$$2^4\theta \leq m_{2^n}(S \otimes S) = 2 \cdot 14 = 28,$$

but $\theta = 3$. The proofs for A_4 and B_4 are similar. ∎

§6.4 Rational Cohomology

We have thus reduced the computation of

$$H^1(G(n),\ \Theta^{e}_{I_1} \otimes \cdots \otimes \Gamma^{e}_{I_{14}} \otimes S_R) = \mathrm{Ext}^1_{kG(n)}(k,\ \Theta^{e}_{I_1} \otimes \cdots \otimes \Gamma^{e}_{I_{14}} \otimes S_R),$$

for $G = D_4$,

$$H^1(G(n),\ \Theta_{I_1} \otimes \cdots \otimes \Gamma^{*}_{I_{14}} \otimes S_R) = \mathrm{Ext}^1_{kG(n)}(k,\ \Theta_{I_1} \otimes \cdots \otimes \Gamma^{*}_{I_{14}} \otimes S_R),$$

for $G = A_4$, and

$$H^1(G(n),\ \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R) = \mathrm{Ext}^1_{kG(n)}(k,\ \Theta_{I_1} \otimes \cdots \otimes (\Gamma\sigma)_{I_{14}} \otimes S_R),$$

for $G = B_4$, to the cases where $|I_1 \cup \cdots \cup I_{14}| \leq 3$ and $|R| \leq 1$. We complete the task by using information about Weyl module structure to obtain cohomology for the algebraic group. We will need here the assumptions that $n > 9$ if $G = D_4$, and $n > 10$ if $G = B_4$ and $n > 11$ if $G = A_4$, in order to apply Theorem 7.1 of [7] (i.e., so that the restriction map

$$\mathrm{Ext}^1_G(L(\mu), L(\nu)) \longrightarrow \mathrm{Ext}^1_{kG(n)}(L(\mu), L(\nu))$$

is an isomorphism).

By the result of Chapter 5, we obtain for A_4:
$\mathrm{Ext}^1_G(k,\Theta) \cong \mathrm{Ext}^1_G(k,\Lambda) \cong \mathrm{Ext}^1_G(k,\mu) \cong \mathrm{Ext}^1_G(k,\Delta) \cong \mathrm{Ext}^1_G(k,\Sigma) \cong$
$\mathrm{Ext}^1_G(k,\mathrm{T}) \cong \mathrm{Ext}^1_G(k,\Gamma) \cong 0 (\cong \mathrm{Ext}^1_G(k,\Theta^*) \cong \cdots \cong \mathrm{Ext}^1_G(k,\Gamma^*))$;

$\mathrm{Ext}^1_G(k,\Psi)\cong k, \mathrm{Ext}^1_G(k,\Delta\Lambda_1)\cong k(\cong \mathrm{Ext}^1_G(k,\Delta^*\Lambda_1^*)), \mathrm{Ext}^1_G(k,\Lambda\Theta_1^*)\cong k$ $(\cong \mathrm{Ext}^1_G(k,\Lambda^*\Theta_1)), \mathrm{Ext}^1_G(k,\Upsilon\Theta_1^*)\cong k(\cong \mathrm{Ext}^1_G(k,\Upsilon^*\Theta_1))$;

For D_4, we have:

$\mathrm{Ext}^1_G(k,\Theta^{abc})\cong \mathrm{Ext}^1_G(k,\Lambda^{abc})\cong \mathrm{Ext}^1_G(k,\Delta^{abc})\cong \mathrm{Ext}^1_G(k,\Gamma^{abc})\cong 0$,

$\mathrm{Ext}^1_G(k,\mu)\cong 2k; \mathrm{Ext}^1_G(k,\Psi)\cong k, \mathrm{Ext}^1_G(k,\Psi\mu_1)\cong k, \mathrm{Ext}^1_G(k,\mu\Theta_1^{abc})\cong k$;

and for B_4, we have

$\mathrm{Ext}^1_G(k,\Theta)\cong \mathrm{Ext}^1_G(k,\Lambda)\cong \mathrm{Ext}^1_G(k,\Delta)\cong \mathrm{Ext}^1_G(k,\Gamma)\cong \mathrm{Ext}^1_G(k,\Sigma)\cong$

$\mathrm{Ext}^1_G(k,\sigma)\cong \mathrm{Ext}^1_G(k,\Theta\sigma)\cong \mathrm{Ext}^1_G(k,\Lambda\sigma)\cong \mathrm{Ext}^1_G(k,\Delta\sigma)\cong \mathrm{Ext}^1_G(k,\Gamma\sigma)\cong 0$;

$\mathrm{Ext}^1_G(k,\Theta)\cong k, \mathrm{Ext}^1_G(k,\mu)\cong k, \mathrm{Ext}^1_G(k,\Psi)\cong k, \mathrm{Ext}^1_G(k,\Psi\mu_1)\cong k$;

$\mathrm{Ext}^1_G(k,\mu\sigma_1)\cong k$,

and (for all three groups) $\mathrm{Ext}^1_G(k,\Pi\Omega_1)\cong 0$ for all other choices of Π,Ω, and $\mathrm{Ext}^1_G(k,\Pi\Omega_2)\cong 0$ for all possible choices of Π,Ω

Most of the remaining cohomology groups can be computed assuming only $n>4$ We consider the seperate cases $|I_1\cup\cdots\cup I_{14}|=0,1,2,3$.

i) $I_1\cup\cdots\cup I_{14}=0$

Since $G(n)$ is simple, $\mathrm{Ext}^1_{kG(n)}(k,k)=0$ Also, we have $\mathrm{Ext}^1_{kG(n)}(k,S_R)=0$ for $R\neq 0$ if $G=B_4$, by Lemma 6 3 2. In the cases $G=D_4$ and A_4, it remains to show that $\mathrm{Ext}^1_{kG(n)}(k,S_1)=0$ and apply Galois conjugation to obtain the result for $R=i$ To accomplish this, we show that $\mathrm{Hom}_{kG(n)}(X(S_1,k)\otimes S, S\otimes k)=\mathrm{Hom}_{kG(n)}(X(S_1,k), S\otimes S)=0$, and apply Lemma 6 3 1 as usual This will show that

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_1,k))\le \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{\{0,1\}},S)),$$

which is zero by Lemma 6.3.2.

Now, $S \otimes S$, as a G-module, has a filtration by Weyl modules

$$0 \subseteq M_1 = V(2\rho) \subseteq \cdots M_k = V(\rho) \otimes V(\rho)$$

The fact that $S \otimes S$ has only one composition factor isomorphic to $S_1 = L(2\rho)$ implies that $\mathrm{Hom}_{kG(n)}(X(S_1, k), M_{i+1}/M_i) = 0$ for all i, except possibly $i = 0$. However, $\mathrm{Hom}_{kG(n)}(X(S_1, k), V(2\rho)) = 0$, because $V(2\rho)$ has simple head isomorphic to S_1, no other composition factors of S_1, and $\mathrm{rad}(V(2\rho))$ contains composition factors other than k. (Injectivity of $\mathrm{Ext}^1_G(L(\mu), L(\nu)) \longrightarrow \mathrm{Ext}^1_{kG(n)}(L(\mu), L(\nu))$ for p^n-restricted weights μ, ν implies that $S \otimes S$ has the same radical series considered as a G-module as when considered as an $kG(n)$-module, if $n > 2$, [for $G = D_4$ or A_4].)

ii) $I_1 \cup \cdots \cup I_{14} = \{i\}$.

The cases where $R = \emptyset$ are listed above. Thus, it remains to dispense with the case $R = \{k\}$ (with $k \neq i$). We show that

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_k, A_i)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{\{i,k\}}, \tilde{A}_i))$$

(which is zero by Lemma 6.3.3), again using Lemma 6.3.1:

$$\mathrm{Hom}_{kG(n)}(X(S_k, A_i) \otimes S_i, \tilde{A}_i) \cong \mathrm{Hom}_{kG(n)}(X(S_k, A_i), S_i \otimes \tilde{A}_i) = 0$$

because S_k cannot be a composition factor of $S_i \otimes \tilde{A}_i$ as $i \neq k$ (cf. Tables 2-1 to 2-3, or by 2^n restricted mass considerations).

iii) $I_1 \cup \cdots \cup I_{14} = \{i, j\}$

If $R = \{k\}$, then in the case $G = D_4$, we argue as follows. Relabeling if necessary, without loss of generality, we assume that $j \neq k-1$. We first argue that $S_i \otimes \tilde{A}_i B_j$ has filtration factors resulting from a composition series of $S_i \otimes \tilde{A}_i$ that are of one of the following 3 forms

i) irreducible and not isomorphic to S_k,

ii) of the form $\Pi_i \otimes \Omega_{i+1} \otimes (\Omega'_{i+2} \otimes B_{i+2})$ (with $\Omega, \Omega' \neq \Gamma^{\mathrm{abc}}, S$),

iii) of the form $\Pi_i \otimes (\Omega_{i+1} \otimes B_{i+1}) \otimes \Omega'_{i+2}$

All of the composition factors of $\Omega'_{i+2} \otimes B_{i+2}$ are of the form $\Pi'_{i+2}\Omega''_{i+3}$, so the resulting filtration factors of type (ii) are irreducible if $n > 3$, with $\Omega'' \neq S$. By assumption, $k \notin \{i, j\}$, therefore S_k cannot be a composition factor of type (ii) if $n > 3$. For filtration factors of type (iii), we argue using 2^n-restricted mass:

$$m_{2^n}(S \otimes AB_1) \leq m_{2^n}(S) + m_{2^n}(AB_1) \leq 14 + 11 + 2 \cdot 11 = 47,$$

whereas S_k will have a 2^n restricted mass of at least $4 \cdot 14 = 56$ if $k \notin \{0, 1\}$. This shows that

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_k, A_i B_j)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{\{i,k\}}, \tilde{A}_i B_j))$$

The arguments for A_4 and B_4 are similar. (We assume $n > 4$ for $G = B_4$.)

Next, we argue that

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_{\{i,k\}}, \tilde{A}_i B_j)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{\{i,j,k\}}, \tilde{A}_i \tilde{B}_j))$$

(which is zero by Lemma 6.3.3) via Lemma 6.3.1 by using 2^n-restricted mass. By twisting, we may assume $k = n-1$. We have

$$m_{2^n}(S_j \otimes \tilde{A}_i \tilde{B}_j) \leq m_{2^n}(S_j) + m_{2^n}(\tilde{B}_j) + m_{2^n}(\tilde{A}_i)$$

$$\leq 2^{n-3} \cdot (14+11) + 2^{n-2} \cdot 11 < m_{2^n}(S_{n-1}) < m_{2^n}(S_K),$$

because $j \neq k-1$. Again, the arguments for this step in the cases $G = A_4$ and B_4 are similar.

We are reduced to the case $R = \emptyset$. Now, if $|i-j| > 2$, we have

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(k, A_i B_j)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_i, \tilde{A}_i B_j)),$$

for $G = D_4, A_4$, and B_4: we have that k is not a composition factor of $S_i \otimes \tilde{A}_i B_j$ since all of the filtration factors resulting from composition factors of $S_i \otimes \tilde{A}_i$ are irreducible of the form $\Pi_i \Omega_{i+1} \Omega'_{i+2} B_j$. Similarly,

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_i, \tilde{A}_i B_j)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S_{\{i,j\}}, \tilde{A}_i \tilde{B}_j)).$$

This is shown to be zero by Lemma 6.3.11.

We may assume henceforth that $\{i,j\} = \{0,1\}$ or $\{0,2\}$ by applying Galois conjugation. Those cases were listed at the beginning of the section.

iv) $I_1 \cup \cdots \cup I_{14} = \{i, i+1, i+2\}$.

By twisting, assume $i = 0$. We shall first dispense with this case if $G = D_4$; the argument for A_4 is similar. By Corollary 6.3.10A, we need only consider the case $R = \emptyset$ and $\Theta^a_{I_1} \otimes \cdots \otimes \Gamma^c_{I_{14}}$ Galois conjugate to $\mu\Theta_1^{abc}\Theta_2^{abc}$. By considering 2^n-restricted mass, it is observed that all of the composition factors of $L(\rho) \otimes L([\rho-\delta_2] + 2\delta_s + 4\delta_t)$ are 2^4-restricted, for $s, t \in \{1,3,4\}$. Thus, since the algebraic group cohomology $\mathrm{Ext}^1_G(k, L(\delta_2 + 2\delta_s + 4\delta_t))$ is zero, (and since $\mathrm{Hom}_{kG(n)}(k, S \otimes \Psi\Theta_1^{abc}\Theta_2^{abc}) = 0$,) we have that

$$\mathrm{Hom}_{kG(n)}(X(k, \mu\Theta_1^{abc}\Theta_2^{abc}), S \otimes \Psi\Theta_1^{abc}\Theta_2^{abc}) = 0,$$

if $n > 3$, because of injectivity of $\mathrm{Ext}^1_G(L(\nu), L(\nu')) \longrightarrow \mathrm{Ext}^1_{kG(n)}(L(\nu), L(\nu'))$ for p^n restricted weights ν, ν'. Therefore, by Lemma 6.3.1, we have

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(k, \mu\Theta_1^{abc}\Theta_2^{abc})) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S, \Psi\Theta_1^{abc}\Theta_2^{abc})),$$

which is zero by Lemma 6.3.9A.

Now let $G = B_4$. We dispense with the case $R = \{k\}$ (with $k \not\equiv 0, 1$, or $2 \bmod(n)$) but will apply the the Cline-Parshall-Scott-Van der Kallen result (and assume $n > 10$) to obtain $\mathrm{Ext}^1_{kG(n)}(k, AB_1C_2))$ (i.e., to handle the case $R = \emptyset$). By 2^n restricted mass considerations, we see that S_k cannot be a composition factor of $S \otimes \bar{A}B_1C_2$, that $S \otimes S_k$ cannot be a composition factor of $S_1 \otimes \bar{A}\bar{B}_1C_2$, and finally that $SS_1 \otimes S_k$ cannot be a composition factor of $S_2 \otimes A\hat{B}_1\check{C}_2$. Therefore, by Lemma 6.3.1, we have

$$\dim_k(\mathrm{Ext}^1_{kG(n)}(S_k, AB_1C_2)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(S \otimes S_k, \bar{A}B_1C_2))$$

$$\leq \dim_k(\mathrm{Ext}^1_{kG(n)}(SS_1 \otimes S_k, \bar{A}\bar{B}_1C_2)) \leq \dim_k(\mathrm{Ext}^1_{kG(n)}(SS_1S_2 \otimes S_k, \bar{A}\bar{B}_1\check{C}_2)),$$

which is zero by Lemma 6.3.6.

CHAPTER 7

FINAL REMARKS

We have computed all of the extensions for the simple modules over the algebraic groups and all of the extensions of the trivial module for all but finitely many of the finite groups for the groups of Lie type A_4, B_4, C_4 and D_4 over an algebraically closed field of characteristic two. One of the main observations that we make is that the extensions for the simple modules over the algebraic group are completely determined by the situation over the second Frobenius kernel, in other words, all of the modules $\mathrm{Ext}^1_G(L(\mu), L(\nu))$ for all dominant weights μ, ν are determined once they are known for the 2^2 restricted weights. In some sense we may phrase this by saying that all of the 1-cohomological information for these algebraic groups is completely determined by knowledge of the action on the simple modules by the differential operators of order ≤ 2

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BIOGRAPHICAL SKETCH

Michael Francis Dowd was born on August 23, 1958, in Providence, Rhode Island. He received a bachelor's degree in mathematics from Harvard College in 1980, and a master's degree in mathematical science from the University of Central Florida in 1986. He served in the United States Navy from 1981 to 1987 as an instructor at the Naval Nuclear Power School in Orlando, Florida. His research interests include representation theory, Lie group and Lie algebra theory, and homological algebra.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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